National Institute of Science Education and Research

Physical Insights into Random Matrix Spectra

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What is a Random Matrix?

A matrix with some or all of its elements as random variables is called a random matrix.

Gaussian Ensembles

Consider the $N \times N$ dimensional matrix A

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots \\ A_{21} & A_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

where $A_{ij} \sim N(0, 1)$ are independent of each other. The eigen values of this matrix need not be symmetric.

Let's symmetrize A...

Gaussian Ensemble

On Symmetrisation...

$$H = \begin{bmatrix} A_{11} & \frac{A_{12}+A_{21}}{2} & \cdots \\ \frac{A_{21}+A_{12}}{2} & A_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Joint Probability Density Function of upper triangular Matrix Elements

$$p[H] = \prod_{i=1}^{N} \left[\exp\left(-H_{ii}^2/2\right) / \sqrt{2\pi} \right] \prod_{i<j} \left[\exp\left(-H_{ij}^2\right) / \sqrt{\pi} \right]$$

Characterizing Features of Gaussian Ensembles

Rotational Invariance

On the similarity transformation $H \to H' = U^{-1}HU$, the functional form of jpdf of the ensemble remains invariant.

$$p[H'] = p[H]$$

where U is a unitary(orthogonal) matrix; if matrix elements of H are complex(real) valued, and hence H is said to be in Gaussian Unitary(Orthogonal) ensemble or GUE(GOE).

Independence

The matrix elements of upper-half triangle are independent of each other.

How do the eigen values of GOE ensemble behave?

Consider 2×2 GOE Random Matrix

$$H_S = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$$

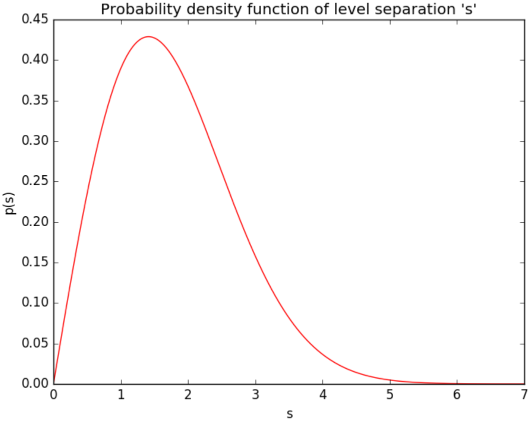
$x_1, x_2 \sim N(0,1)$ and $x_3 \sim N(0, 1/2)$.

and observe the separation between its two eigen values$\lambda_{\pm}$.

$$s = \lambda_+ - \lambda_-$$

The probability density of the separation s, between the eigen values implies level-repulsion and confinement.

$$p(s) = \frac{s}{2} e^{-s^2/4}$$



Joint probability density of eigenvalues of N-dimensional Gaussian Ensembles

$$p\left(x_1, \ldots, x_N\right) = \frac{1}{\mathcal{Z}_{N,\beta}} e^{-\frac{1}{2}\sum_{i=1}^{N} x_i^2} \prod_{j<k} \left|x_j - x_k\right|^{\beta}$$

where $\mathcal{Z}_{N,\beta}$ is the normalisation constant. β is the number of real entries required for a matrix element[1].

Confinement factor

$$e^{-\frac{1}{2}\sum_{i=1}^{N} x_i^2}$$

Level-Repulsion factor

$$\prod_{j<k} \left|x_j - x_k\right|^{\beta}$$

How will the gap between random variables behave if ...

$X_1, X_2 \dots X_N$ were i.i.d. with pdf $p(x)$, instead of being the eigenvalues of $N \times N$ Gaussian Random Matrix.

For a large but finite N

- The number of Random Variables in an interval dx at x is $Np(x)dx$.
- The separation between the random variable goes as

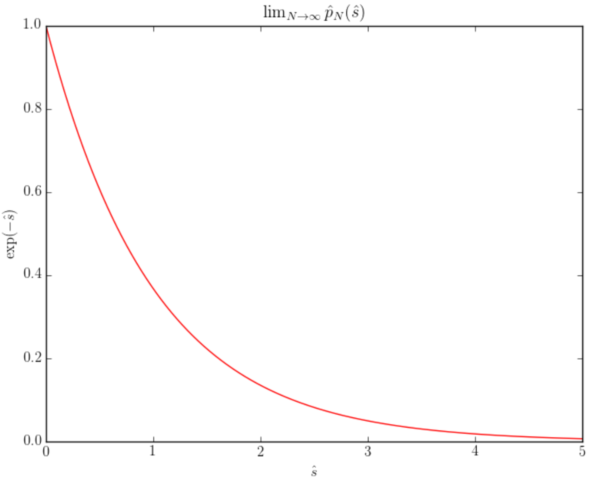
$$s \sim \frac{dx}{Np(x)dx} \sim \frac{1}{Np(x)}$$

- Define the variable $\hat{s} = sNp(x)$.

What does one expect about the probability density of $\hat{s}$?

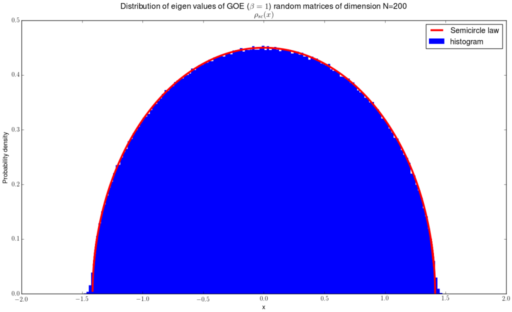
But!!!

i.i.d. random variables are attractive.

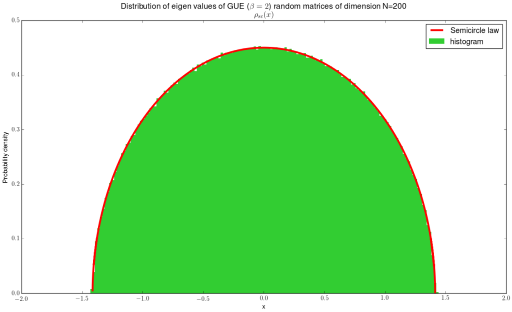


Now, let's observe the distribution of eigenvalues in the limit $N \to \infty$

GOE


Distribution of eigen values of GOE ($\beta = 1$) random matrices of dimension N=200
$\rho_{sc}(x)$
Semicircle law
histogram
Probability density
x

GUE


Distribution of eigen values of GUE ($\beta = 2$) random matrices of dimension N=200
$\rho_{sc}(x)$
Semicircle law
histogram
Probability density
x

The shape of the Histogram

Wigner's Semicircle law

$$p(x) = \frac{1}{\pi}\sqrt{2 - x^2}$$

Can this be analytically derived?

Coulomb Gas Interpretation

$$\mathcal{Z}_{N,\beta} = C_{N,\beta} \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j e^{-\frac{\beta}{2} N \sum_{i=1}^{N} x_i^2} \prod_{j<k} \left| x_j - x_k \right|^{\beta}$$

$$= C_{N,\beta} \int_{\mathbb{R}^N}^{N} \prod_{j=1}^{N} dx_j e^{-\beta N^2 \mathcal{V}[\mathbf{x}]}$$

where $C_{N,\beta}$ is a constant. $\mathcal{V}[\mathbf{x}]$ is given by

$$\mathcal{V}[\boldsymbol{x}] = \frac{1}{2N} \sum_{i} x_i^2 - \frac{1}{2N^2} \sum_{i \neq j} \ln \left| x_i - x_j \right|$$

Taking Continuum Limit. . .

The Continuum Limit

$$\mathcal{Z}_{N,\beta} \approx C_{N,\beta} \int \mathcal{D}[n(x)] \int_{\mathbb{R}} d\kappa e^{-\beta N^2 \mathcal{S}[n(x),\kappa] + \mathcal{O}(N)}$$

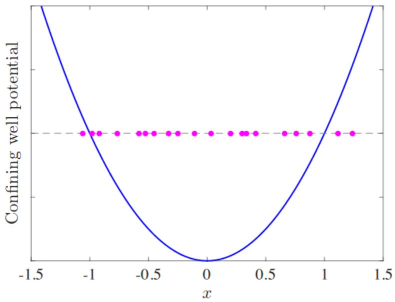
where the action $\mathcal{S}[n(x), \kappa]$ is

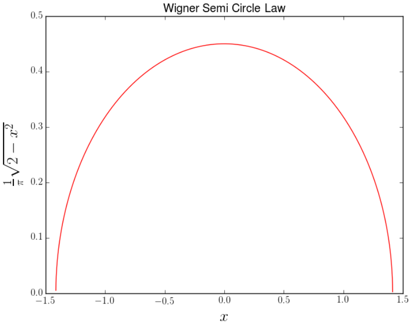
$$\mathcal{S}[n(x), \kappa] = \mathcal{F}_0[n(x)] - \kappa \left(\int dx n(x) - 1 \right)$$

and the free energy $\mathcal{F}_0[n(x)]$ is

$$\mathcal{F}_0[n(x)] = \frac{1}{2} \int dx n(x) x^2 - \frac{1}{2} \iint dx dx' n(x) n\left(x'\right) \ln\left|x - x'\right|$$

On Minimising Free Energy. . .





The distribution of Eigenvalues follows

$$n(x) = \frac{1}{\pi}\sqrt{2 - x^2}$$

Consider the eigenvalues of N-dimensional Random Matrix in equillibrium.

The Equillibrium Configuration

The equillibrium configuration is the one that maximises the joint probability density[2].

$$\sum_{j \neq i} \frac{1}{x_i - x_j} - N x_i = 0 \tag{1}$$

The solution to this system of equations are given by the zeros of the Hermite polynomial H_N,

$$H_N\left(\sqrt{N} x_i\right) = 0 \tag{2}$$

Remark:Roots of hermite polynomials give the nodes of 1-D harmonic oscillator wave function

Normal Modes of Oscillation

$$\delta y_i^{(1)} = \frac{1}{N^{1/2}}$$

$$\delta y_i^{(2)} = \left(\frac{2}{N-1}\right)^{1/2} x_i$$

$$\delta y_i^{(3)} = \left(\frac{N-1}{N(N-2)}\right)^{1/2} \left(1 - \frac{2N}{N-1} x_i^2\right)$$

$$\delta y_i^{(4)} = \left(\frac{2(2N-3)^2}{(N-1)(N-2)(N-3)}\right)^{1/2} \left(x_i - \frac{2N}{2N-3} x_i^3\right)$$

δy_i is the displacement of X_i about its equillibrium position x_i[3].

The problem in QCD

QCD Lagrangian: $\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}\left(i\gamma_\mu D^\mu - m\right)\psi$

where $D^\mu = \partial^\mu - igA^\mu(x)$

Problem

- QCD is an extremely difficult problem to handle analytically.
- Perturbation method does not work.

Therefore, lattice QCD is used.

Ongoing Research

Get the spectrum of QCD Action in differnt discretisation schemes.

Attempts and Target

- Wilson Fermions[4]. $\leftarrow$ Attempted but failed
- Some other examples: Kogut-Susskind Fermions, Domain Wall fermions, Overlap fermions, Twisted-mass fermions
- Karsten-Wilczek fermions $\leftarrow$ Envisaged application

References

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Thank You