Fluctuations over a Bjorken Background

A project report submitted by

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Abstract

The relativistic equations of hydrodynamics are the conservation of energy-momentum and the conserved charges. Bjorken flow is an exact solution of ideal hydrodynamics which is a good model to describe relativistic heavy ion collision. Also it is also boost invariant solution. Therefore we chose it as the background and studied the evolution of perturbation over it. Differential equations describing the same are derived upto first order in perturbation. Assuming rapidity independence of the solution we get a diverging solution for the perturbation, thus limiting the validity of the differential equations obtained. Moreover the background should have a small velocity. Also theoretical constraints of preserving the length of four velocity vector is discussed.

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Chapter 1

Introduction

Apart from it's applications in non-relativistic regime, hydrodynamics has been quite useful in describing the flow of matter formed when heavy nuclei collide with each other at a speed approaching to that of light[1, 2]. This motivated theoretical investigations into relativistic hydrodynamics.

Amongst many other research topic, study of perturbations has been quite active owing to the fact that one cannot avoid them in real situations. Deviations from ideal models are inevitable and we must have how they affect the system. Moreover, studying perturbations gives important physical insights. A naive analysis of perturbation over a uniform background gives reveals the acausality of relativistic hydrodynamics[3].

This motivated to explore the perturbation on a different background. The physics of heavy ion collision have been influenced by Bjorken's flow[4, 5]. Even though it is very difficult to get an analytic solution, successful attempts has been made to obtain one has been made for some particular cases. Bjorken's solution neglects transverse motion of fluid elements, while many of these solutions happen to generalize the Bjorken's solution to a more generic context by relaxing the assumption made by Bjorken[6, 7].

At very high energies, of the order of 100 GeV, the number of particles produced is almost equal to that of anti-particles. Hence one generally neglects the chemical potential corresponding to any conserved charges in heavy ion collision at high energy regime. Nevertheless, there is a small baryon number density which evolve with time and rapidity[8]. Also the evolution of fluctuations in fluid four-velocity and energy-density has been studied, with viscous effects taken into account.

In this project we discuss the perturbation over Bjorken background neglecting the baryon number density. This restricts our treatment to high energy regime only. Furthermore we consider only the ideal hydrodynamics and apply perturbations to the Bjorken four-velocity and obtain differential equations which describe their evolution.

This work is organised as follows. In the second chapter we introduce the basic equations of Ideal relativistic hydrodynamics. Bjorken flow and it's characteristics have been discussed in the following chapter. Differential equations for the evolution of perturbations have been derived. Analytic solutions have been obtained making the assumption of rapidity independence of the perturbations. Also the validity of obtained solution and theoretical constraints on perturbation and background have been discussed.

Chapter 2

Ideal Relativistic Hydrodynamics

The[**?**] basic equations are the conservation of energy-momentum

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.1}$$

where $T^{\mu\nu}$ is the energy-momentum tensor of the system, and the current conservation

$$\partial_\mu N_i^\mu = 0 \tag{2.2}$$

where N_i^ν is the i^{th} conserved current, such as the baryon current in heavy ion collision, electric charges etc.

We shall use the flat space time metric $g^{\mu\nu} = diag(1, -1, -1, -1)$. The four velocity satisfies the relation

$$u^\mu u_\mu = 1 \tag{2.3}$$

We introduce the tensor $\Delta^{\mu\nu}$ which is defined as

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \tag{2.4}$$

The input to these equations are the constitutive relations which relates $T^{\mu\nu}$ and N_i^μ to the hydrodynamic variables u^μ, ϵ, p, n. Here ϵ, p and n are the energy density, pressure and number density of conserved charges the system. Finally we need the equation of state, which is a relation between ϵ, p and n, so as to close the system of equations. The energy momentum tensor for ideal hydrodynamics is given by

$$T^{\mu\nu} = (\epsilon + p)u^{\mu\nu} - pg^{\mu\nu} \tag{2.5}$$

Projecting eqn. 2.5 parallel to the fluid velocity u^μ we obtain

$$u_\nu \partial_\mu T^{\mu\nu} = u_\nu \partial_\mu (\epsilon u^\mu u^\nu - p\Delta^{\mu\nu})$$

Using the fact that $u^\mu u_\mu = 1$ and hence $u_\mu \partial_\nu u^\nu = 0$, and simple algebraic manipulation leads to

$$u_\nu \partial_\mu T^{\mu\nu} = u^\mu \partial_\mu \epsilon + \epsilon(\partial_\mu u^\mu) + \epsilon u_\nu u^\mu \partial_\mu u^\nu - p u_\mu \partial_\nu \Delta^{\mu\nu} \tag{2.6}$$

This implies

$$u_\nu \partial_\mu T^{\mu\nu} = (\epsilon + p)\partial_\mu u^\mu + u^\mu \partial_\mu \epsilon \tag{2.7}$$

The other projection gives

$$\Delta^\alpha_\nu \partial_\mu T^{\mu\nu} = \epsilon u^\mu \Delta^\alpha_\nu \partial_\mu u^\nu - \Delta^{\mu\alpha}(\partial_\mu p) + p u^\mu \Delta^\alpha_\nu \partial_\mu u^\nu \tag{2.8}$$

Arranging the terms in above equation leads to

$$\Delta^\alpha_\nu \partial_\mu T^{\mu\nu} = (\epsilon + p) u^\mu \partial_\mu u^\alpha - \Delta^{\mu\alpha} \partial_\mu p \tag{2.9}$$

Introducing short hand notations

$$D = u^\mu \partial_\mu \tag{2.10}$$

and

$$\nabla^\alpha = \Delta^{\mu\alpha} \partial_\mu \tag{2.11}$$

Using above notations in the equation of perpendicular (eqn. 2.9) and parallel (eqn. 2.7) projection, eqn. 2.5 is decomposed to following two equations

$$D\epsilon + (\epsilon + p)\partial_\mu u^\mu = 0 \tag{2.12}$$

and

$$(\epsilon + p) D u^\alpha - \nabla^\alpha p = 0 \tag{2.13}$$

Equations 2.12 and 2.13 are the fundamental equations of relativistic ideal fluid.

Chapter 3

Bjorken Flow

According to Bjorken's picture of the fluid flow, the matter at a distance z after a time t flows at a velocity $v^z = z/t$. Neglecting the transverse flow ($v^x = v^y = 0$) the four velocity of the matter is

$$u^\mu = \gamma(1, 0, 0, v^z) = \frac{1}{\sqrt{1 - (z/t)^2}}(1, 0, 0, z/t) = (t/\tau, 0, 0, z/\tau)$$

where $\tau = \sqrt{t^2 - z^2}$. Since $u^\mu u_\mu = 1$ and we have only two non vanishing four-velocity components, it is convinient to work in the 1+1 dimension with coordinates $x^\mu = (t, z)$, with the metric $g^{\mu\nu} = diag(1, -1)$ velocity $u^\mu = (t/\tau, z/\tau)$.

It is convinient to work in the coordinates of proper time τ and rapidity η, where

$$\eta = \frac{1}{2} \ln \frac{t + z}{t - z} \tag{3.1}$$

The expressions for η and τ can be inverted to get

$$t = \tau \cosh \eta \tag{3.2}$$

$$z = \tau \sinh \eta \tag{3.3}$$

It can be shown that a lorentz boost of velocity v corresponds to a shift in the value of η by $\tanh^{-1} v$. Bjorken's notion of boost invariance corresponds to vanishing of η-component of fluid velocity.

$$u^\tau = u^t \partial_t \tau + u^z \partial_z \tau = \cosh^2 \eta - \sinh^2 \eta = 1$$

$$u^\eta = u^t \partial_t \eta + u^z \partial_z \eta = -u^t \frac{\sinh \eta}{\tau} + u^z \frac{\cosh \eta}{\tau} = 0$$

Vanishing η-component of fluid velocity renders the fluid parameters ϵ, T, p etc. to be independent of η, and therefore these are unchanged under a lorentz transformation.

The derivatives w.r.t t and z in terms of that of τ and η obtained using coordinate transformation are as follows

$$\partial_t = \cosh\eta\partial_\tau - \frac{\sinh\eta}{\tau}\partial_\eta \tag{3.4}$$

$$\partial_z = -\sinh\eta\partial_\tau + \frac{\cosh\eta}{\tau}\partial_\eta \tag{3.5}$$

The derivative $u^\mu\partial_\mu$ becomes derivative w.r.t. τ.

$$\begin{aligned} u^\mu\partial_\mu &= u^t\partial_t + u^z\partial_z \\ &= \cosh\eta(\cosh\eta\partial_\tau - \frac{\sinh\eta}{\tau}\partial_\eta) + \sinh\eta(-\sinh\eta\partial_\tau + \frac{\cosh\eta}{\tau}\partial_\eta) \\ &= \partial_\tau \end{aligned}$$

Similarly $\partial_\mu u^\mu = 1/\tau$. With this eqn. 2.12 becomes

$$\partial_\tau\epsilon + \frac{\epsilon + p}{\tau} = 0 \tag{3.6}$$

and eqn. 2.13 becomes

$$u_\nu\partial_\tau p + (\epsilon + p)\partial_\tau u_\nu - \partial_\nu p = 0 \tag{3.7}$$

Even without using the equation of state the proper time evolution of entropy can be deduced by using thermodynamic relations $\epsilon = Ts - p$ and $d\epsilon = Tds$, where s is the entropy density. Using this the eqn, 3.6 becomes

$$\partial_\mu(su^\mu) = 0 \tag{3.8}$$

This simplifies to

$$\partial_\tau s + s/\tau = 0 \tag{3.9}$$

Hence we obtain the proper time evolution of entropy density as

$$s = s(\tau_0)\frac{\tau_0}{\tau} \tag{3.10}$$

Using the equation of state

$$\epsilon = 3p$$

we obtain the evolution of energy density from eqn. 3.6

$$\epsilon = \epsilon_0\left(\frac{\tau_0}{\tau}\right)^{4/3} \tag{3.11}$$

The equation of state gives the proper time evolution of pressure p. From thermodynamic relations we obtain T in terms of ϵ, p and s

$$T = \frac{\epsilon + p}{s} = \frac{4}{3}\frac{\epsilon}{s} = \frac{4}{3}\frac{\epsilon_0}{s_0}\left(\frac{\tau_0}{\tau}\right)^{1/3}$$

Chapter 4

Perturbations over Bjorken Flow Background

4.1 Equations for evolution of Perturbation

In this chapter we shall derive the equations governing evolution of perturbations upto linear order. To do so we split the background and perturbation as

$$\epsilon = \bar{\epsilon} + \delta\epsilon \tag{4.1}$$

and

$$u^\mu = \bar{u}^\mu + \delta u^\mu \tag{4.2}$$

where $\bar{\epsilon}$ and $\bar{u}^\mu$ are the background energy density and four velocity while $\delta\epsilon$ and δu^μ are the perturbation in corresponding quantities. Using these for quantities the equation 2.12 becomes

$$\begin{aligned} \bar{u}^\mu \partial_\mu \bar{\epsilon} + (\bar{\epsilon} + \bar{p})\partial_\mu \bar{u}^\mu + \delta u^\mu \partial_\mu \bar{\epsilon} + \bar{u}^\mu \partial_\mu \delta\epsilon \\ + (\bar{\epsilon} + \bar{p})\partial_\mu \delta u^\mu + (\delta\bar{\epsilon} + \delta\bar{p})\partial_\mu \bar{u}^\mu = 0 \end{aligned} \tag{4.3}$$

and equation 2.13 becomes

$$\begin{aligned} (\bar{u}_\nu \delta u^\mu + \bar{u}^\mu \delta u_\nu)\partial_\mu \bar{p} + \bar{u}_\nu \bar{u}^\mu \partial_\mu \delta p + (\bar{\epsilon} \\ + \bar{p})\bar{u}^\mu \partial_\mu \delta u_\nu + (\bar{\epsilon} + \bar{p})\delta u^\mu \partial_\mu \bar{u}_\nu + (\delta\epsilon + \delta p)\bar{u}^\mu \partial_\mu \bar{u}_\nu - \partial_\nu \delta p = 0 \end{aligned} \tag{4.4}$$

Note that the free index in this equation ν runs over four components of four-velocity vector. Hence we have five differential equations for the perturbation in energy density and that in four components of velocity vector.

4.2 Bjorken Flow as Background

Now we choose the Bjorken flow as the background and neglect the transverse motion. That is, (in cartesian coordinates) $\delta u^x = \delta u^y = 0$. Thus the non-vanishing perturbations are δu^t and δu^z. It is convenient to work in the coordinates (τ, η).

The quantities $\delta u^\mu \partial_\mu$ and $\partial_\mu \delta u^\mu$ can be evaluate to

$$\partial_\mu \delta u^\mu = \partial_\tau \delta u^\tau + \partial_\eta \delta u^\eta + \frac{\delta u^\tau}{\tau} \tag{4.5}$$

and

$$\delta u^\mu \partial_\mu = \delta u^\tau \partial_\tau + \delta u^\eta \partial_\eta \tag{4.6}$$

Also we shall use the equation of state $\bar{\epsilon} = 3\bar{p}$, and $\delta\epsilon = 3\delta p$. Using these the equation 4.3 becomes

$$\partial_\tau \delta\epsilon + \frac{4}{3}\frac{\delta\epsilon}{\tau} + \frac{4}{3}\epsilon_0 \left(\frac{\tau_0}{\tau}\right)^{4/3} (\partial_\eta \delta u^\eta + \partial_\tau \delta u^\tau) = 0 \tag{4.7}$$

and from equation 4.4 one obtains

$$\begin{aligned} &\bar{u}_\nu \delta u^\mu \partial_\mu \bar{\epsilon} + \delta u_\nu \partial_\tau \bar{\epsilon} + \bar{u}_\nu \partial_\tau delta\epsilon + 4\bar{\epsilon}\partial_\tau \delta u_\nu \\ &\qquad + 4\bar{\epsilon}\delta u^\mu \partial_\mu \bar{u}_\mu + 4\delta\epsilon \partial_\tau \bar{u}_\nu - \partial_\nu \delta\epsilon = 0 \end{aligned}$$

Using 4.5 and 4.6 the above expression simplifies to

$$\begin{aligned} &-\frac{4}{3}\epsilon_0 \tau_0^{4/3} \tau^{-7/3} (\delta u^\tau \bar{u}_\nu + \delta u_\nu) + \bar{u}_\nu \partial_\tau \delta\epsilon \\ &\qquad + 4\epsilon_0 \tau_0^{4/3} \tau^{-4/3} \partial_\tau \delta u_\nu - \partial_\nu \delta\epsilon = 0 \end{aligned} \tag{4.8}$$

where ν varies over τ and η. For $\nu = \tau$, on a Bjorken background($\bar{u}_\tau = 1$), one obtains

$$\frac{2}{3}\frac{\delta u^\tau}{\tau} + \frac{\partial \delta u^\tau}{\partial \tau} = 0 \tag{4.9}$$

This differential equation is satisfied by

$$\delta u^\tau = \delta u^\tau(\tau_0, \eta) \left(\frac{\tau}{\tau_0}\right)^{2/3} \tag{4.10}$$

Using this solution in equation 4.7 we get

$$\partial_\tau \delta\epsilon + \frac{4}{3}\bar{\epsilon}\partial_\eta \delta u^\eta + \frac{4}{3}\frac{\delta\epsilon}{\tau} + \frac{8}{9}\delta u^\tau(\tau_0, \eta) \left(\frac{\tau}{\tau_0}\right)^{2/3} = 0 \tag{4.11}$$

For $\nu = \eta$, we obtain the following differential equation from 4.8

$$\frac{4}{3}\epsilon_0 \tau_0^{4/3} \tau^{-7/3} \delta u^\eta + 4\epsilon_0 \tau_0^{4/3} \tau^{-4/3} \partial_\tau \delta u^\eta - \partial_\eta \delta\epsilon = 0 \tag{4.12}$$

Thus we obtian two differential equations for two unknowns $\delta\epsilon$ and δu^η. These equations can be solved numerically to obtain the solutions for perturbation in energy density and η component of velocity.

To get an analytic solution we again assume η-independence so that terms with ∂_η vanish and we obtain a system of differential equations in first derivative of τ. The system of equations 4.11 and 4.12 then become

$$\begin{aligned} \partial_\tau \delta u^\eta - \frac{1}{3}\frac{\delta u^\eta}{\tau} &= 0 \\ \partial_\tau \delta\epsilon + \frac{4}{3}\frac{\delta\epsilon}{\tau} + \frac{8}{9}\delta u^\tau(\tau_0, \eta)\tau^{-1/3} &= 0 \end{aligned} \tag{4.13}$$

The system of equations is solved by

$$\delta u^\eta = \delta u^\eta(\tau_0)\left(\frac{\tau}{\tau_0}\right)^{1/3} \tag{4.14}$$

and

$$\delta\epsilon = \delta\epsilon_0 \left(\frac{\tau}{\tau_0}\right)^{2/3} \tag{4.15}$$

We thus found a diverging solution for the perturbation. In the next section we discuss some of the theoretical problems concerned with solutions obtained.

4.3 Constraints on the Solution obtained

The solution obtained was based on the assumption that the perturbation were small enough for neglecting terms of order gerater than linear terms. Since the perturbations are strictly increasing with proper-time, the differential equations which determines their evolution becomes invalid.

The requirement of $u^\mu u_\mu = 1$ even after adding the perturbations imposes another constraint discussed below. Therefore constraints cannot be added arbitrarily. Rather one needs to respect the constraint

$$(u^\mu + \delta u^\mu)(u_\mu + \delta u_\mu) = 1 \tag{4.16}$$

Expanding the left hand side of above expression and using $u^\mu u_\mu = 1$ gives us an equation in the constraints

$$u^\mu \delta u_\mu = -\frac{1}{2}\delta u^\mu \delta u_\mu \sim O(\delta^2) \tag{4.17}$$

The left hand side of above equation gives

$$u^\mu \delta u_\mu = \delta u_\tau \tag{4.18}$$

which is a linear order term. This statement means that the requirement of $u^\mu u_\mu$ to be invariant even after addition of perturbation impoese the condition that linear order terms to be of same order of magnitude as that of second order terms. This will happen only if u^μ itself is very small. This implies that the perturbations can be studied only if the background velocity is small enough for condition 4.18 to be satisfied.

Chapter 5

Conclusion

The perturbation in the velocity field of a background flow affects fluid properties like energy density, temperature, pressure etc. The thermodynamic properties of the fluid are related by the equation of state while the constitutive relations relates them to the energy-momentum tensor and four-current. The conservation of energy, momentum and conserved charges gives the equations for evolution of the velocity fields, energy density. Other thermodynamic perturbations are determined by using thermodynamics.

The Bjorken flow being a good model for describing flow of matter formed at relativistic heavy ion collision, is a good candidate to be choosen as a background. The Bjorken's notion of boost invariance lies in the fact that the evolution of thermodynamic parameters are rapidity independent. The energy density, pressure, temperature and entropy are vary inversely as a power of proper time. Thus for arbitrarily small value of proper time, these quantities tend to be infinite.

The differential equations for evolution of perturbation of four velocity is derived upto the first order in deviation from the background. Neglecting perturbations transverse to the direction of flow of heavy ion, the perturbations in rapidity and proper time component of four velocity diverges as one-third and two-third power of proper time respectively.This restricts the validity of differential equations derived for determining the evolution of perturbation.

Constraining the four velocity to be of unit length after adding the perturbation $(u^\mu + \delta u^\mu)(u_\mu + \delta u_\mu) = 1$, imposes the condition that the linear order perturbation should be of the magnitude of a second order term. This happens only for small background velocity.

In our further work we would like to explore the properties of differential equations we derived. The crucial questions concern the conservation of magnitude of four-velocity vector after the evolution. Also we would like to see if there exist a lorentz transformation connecting the initial velocity vector to the evolved one.

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