6th Semester Project

Acausality Problem in Relativistic Fluid Dynamics

Renu Raman Sahu Roll Number : 1511070

Under the guidance of
Dr. Victor Roy

School Of Physical Sciences
National Institute of Science Education and Research
Jatni, Khurda 752050, Odisha, India

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Abstract

Fluidity is an ubiquitous property encountered frequently in many physical systems and at various energy scales. In the non-relativistic limit, there is a well understood formalism to understand fluid properties. In the present project,the formalism to understand ideal fluids, which do not dissipate, is discussed. The applications of these equations is illustrated for different cases such as hydrostatics, the case of incompressible fluids etc. In order to take dissipation into account Navier-Stokes equation is used, which is an extension of the Euler equation. It is shown that the the kinetic energy density of the fluid decreases with time as a result of the dissipative term in Navier-Stokes equation. At higher energy scales the relativistic effects needs to be taken into account. Equations for relativistic ideal fluid dynamics have been discussed. It is shown that it reduces to the continuity equation and Euler equation in the non-relativistic limit. But generalising Navier-Stokes Equation leads to the problem of acausality. There are different approaches to incorporate causality into the theory of relativistic hydrodynamics. The results of Maxwell-Cattaneo approach and Muller-Israel-Stewart approach are discussed, and it is shown how the equations obtained by these methods respects causality.

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Chapter 1

Ideal Fluids

1.1 Introduction

Fluids are substances that flow. Hence it's dynamics are concerned with substances that can flow namely liquids and gases. A fluid is regarded as a continuous medium. The mathematical treatment of fluids assumes that even an infinitesimal element contains large enough number of constituent particles which has same bulk properties as that of a considerable volume. A fluid particle will be referred to as this volume element rather than a molecule constituting the fluid.

Since fluid occupy certain spatial volume and is continuous, a coordinate system is set up to keep track of the dynamics exhibited by the fluid. Corresponding to each spatial coordinate we assign some physical quantity to describe the state of the fluid. The state of the fluid is determined by the velocity distribution function $\vec{v} = \vec{v}(x, y, z, t)$ and any two of the thermodynamic quantities such as pressure $p(x, y, z, t)$ and fluid mass density $\rho(x, y, z, t)$ at each coordinate (x, y, z, t). Hence there are five independent degrees of freedom. The parameters that completely determine the state of the fluid are v_x, v_y, v_z, p and ρ. Solving a fluid dynamics problem boils down to finding these parameters for different situations of interest. Once we know the state of the fluid at a given space-time coordinate we need to find the state at some other coordinate.

1.2 Continuity Equation

The continuity equation is a manifestation of conservation of matter. It states that the rate at which mass flows out of a volume is same as the rate of change of density of the volume. The amount of matter in a volme V

is given by $\int \rho dV$. Mass of fluid flowing out of the volume per unit time through a surface element $\vec{dS}$ of the volume V is given

$$\oint_S \rho\vec{v}\vec{dS}$$

where S is the surface enclosing the volume V. The decrease in mass per unit time in the volume is

$$-\frac{\partial}{\partial t}\int \rho dV$$

Conservation of mass leads us to equate the decrease in mass per unit time in the volume to the total amount of fluid going out of the volume per unit time. Hence,

$$\begin{aligned} -\frac{\partial}{\partial t}\int \rho dV &= \oint \rho\vec{v}\vec{dS} \\ \implies -\int_V \frac{\partial}{\partial t}\rho dV &= \int_V \vec{\nabla}(\rho\vec{v})d\vec{V} \\ \implies \int_V \left(\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho\vec{v})\right) dV &= 0 \end{aligned}$$

For a non zero volume, the above integral can be satisfied only if the integrand vanishes. Defining $\vec{J} = \rho\vec{v}$ we get

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{J} = 0 \tag{1.1}$$

$\vec{J}$ is also known as mass flux density. It's magnitude represents the mass of fluid flowing per unit time per unit area perpendicular to the velocity of the fluid. It's direction is the direction in which the fluid is flowing.

Eqn. 1.1 states that the mass is conserved at each point of the space. Hence this is a local conservation equation. Local conservation of mass also implies it's global conservation. This can be seen by integrating the continuity equation over a volume large enough such that it's surface is at infinite distance from the fluid system.

$$\begin{aligned} \int_V \frac{\partial\rho}{\partial t}dV + \int_V \vec{\nabla}\vec{J}dV = 0 \\ \int_V \frac{\partial\rho}{\partial t}dV + \oint \vec{J}\vec{dS} = 0 \end{aligned}$$

The surface integral vanishes as no fluid is passing through it. This implies that

$$\int_V \frac{\partial\rho}{\partial t}dV = 0$$

$$\frac{\partial}{\partial t}\int_V \rho dV = 0 \tag{1.2}$$

This is the equation for global conservation of mass.

1.3 Euler Equation

Euler equation is a manifestation of Newton's second law of motion. Hence it describes the time evolution of fluid state parameters. Consider a surface element $d\vec{S}$. The force $d\vec{F}$ acting on it is $d\vec{F} = pd\vec{S}$. The force acting on a volume V is the vector addition of the force $\overrightarrow{dF}$ acting on each element of the surface bounding the volume. Hence

$$\overrightarrow{F} = -\oint_S p\overrightarrow{dS} = -\int_V \overrightarrow{\nabla} pdV \tag{1.3}$$

Therefore the fluid surrounding any volume element dV exert a force $-\overrightarrow{\nabla}pdV$ on the element. Hence force per unit volume will be $-\overrightarrow{\nabla}p$. By Newton's law $\overrightarrow{F} = md\vec{v}/dt$ where m is the mass of the volume element. So the force per unit volume is $\rho d\vec{v}/dt$. So the equation of motion is

$$\rho\frac{d\vec{v}}{dt} = -\overrightarrow{\nabla}p \tag{1.4}$$

The velocity $\vec{v}(x, y, z, t)$ is a vector field that depends on spatial and temporal coordinates. Hence any change in the coordinates will affects the velocity field as

$$\begin{aligned} \frac{d\vec{v}}{dt} =& \frac{\partial\vec{v}}{\partial x}\frac{dx}{dt} + \frac{\partial\vec{v}}{\partial y}\frac{dy}{dt} + \frac{\partial\vec{v}}{\partial z}\frac{dx}{dt} + \frac{\partial\vec{v}}{\partial t} \\ \implies \frac{d\vec{v}}{dt} =& (\vec{v}\cdot\overrightarrow{\nabla})\vec{v} + \frac{\partial\vec{v}}{\partial t} \end{aligned}$$

Using this in eqn. 1.4 we get

$$(\vec{v}\cdot\overrightarrow{\nabla})\vec{v} + \frac{\partial\vec{v}}{\partial t} = -\frac{1}{\rho}\overrightarrow{\nabla}p \tag{1.5}$$

This is the Euler equation which describes the time evolution of state of the fluid. Note that this equation is completely in terms of the fluid parameters.

If there is a gravitational field then the force per unit volume would be ρg. The Euler equation then becomes

$$(\vec{v}\cdot\vec{\nabla})\vec{v}+\frac{\partial\vec{v}}{\partial t}=-\frac{1}{\rho}\vec{\nabla}p+\vec{g} \tag{1.6}$$

This equation useful for many real life situations as we experience a non zero acceleration due to gravity. Thermodynamic and viscosity considerations are not taken into account in deriving the above equations. Such fluids in which thermal conductivity and viscosity are unimportant are called ideal fluids.

We say the motion of the fluid to be adiabatic if there is no heat exchange between different parts of the fluid. In adiabatic motion, the entropy of any fluid element remains constant in the course of it's spatial motion. Since the entropy of each fluid element is constant we require for the motion to be adiabatic

$$\frac{ds}{dt}=0 \tag{1.7}$$

where s is the entropy density(entropy per unit mass). This implies

$$(\vec{v}\cdot\vec{\nabla})s+\frac{\partial s}{\partial t}=0 \tag{1.8}$$

Multiplying eqn. 1.1 with s and eqn. 1.8 with ρ, and then adding, we obtain

$$\begin{aligned}(\rho\vec{v})\vec{\nabla}s+\frac{\partial s}{\partial t}+s\vec{\nabla}(\rho\vec{v})+s\frac{\partial\rho}{\partial t}=0\\ \implies\vec{\nabla}(\rho s\vec{v})+\frac{\partial}{\partial t}(\rho s)=0\end{aligned}$$

Here $\rho s\vec{v}$ is the entropy flux density. Since the entropy per unit volume is constant, the fluid retains the same entropy as it started out. That is the entropy density at any diven time is same as that was there initially. Such motion is called isentropic motion.

Using thermodynamic equations we can eleminate pressure P from the equation of motion, and express it completely in trems of velocity and it's spatial and temporal derivatives. Here we take a digression to look into some equations of thermodynamics. The heat function H, also known as enthalpy, is given by

$$\begin{aligned}H=E+pV\\ \implies dH=dE+pdV+Vdp\end{aligned}$$

Using $dE=TdS-pdV$ we obtain

$$\begin{aligned}dH=&TdS-pdV+pdV+Vdp\\ =&TdS+Vdp\end{aligned}$$

For isentropic process $dS = 0$ which implies $dH = Vdp$. Defining $dw = dH/\delta m$, Dividing above equation by δm we obtain

$$\begin{aligned} \frac{dH}{\delta m} &= T\frac{dS}{\delta m} + \frac{V}{\delta m}dp \\ \implies dw &= Tds + \frac{1}{\rho}dp \end{aligned}$$

For isentropic process it becomes $dw = dp/\rho$. In three dimensions we have

$$\vec{\nabla} w = \frac{1}{\rho}\vec{\nabla} p$$

Hence, the Euler equation 1.5 becomes

$$(\vec{v}\vec{\nabla})\vec{v} + \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} w \tag{1.9}$$

We have from vector analysis

$$\frac{1}{2}\vec{\nabla} v^2 = \vec{v} \times (\vec{\nabla} \times \vec{v}) + (\vec{v} \cdot \vec{\nabla})\vec{v} \tag{1.10}$$

Substituting $\vec{v}) + (\vec{v} \cdot \vec{\nabla})\vec{v}$ in the Euler equation from above equation we get

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}\left(\frac{v^2}{2} - w\right) \tag{1.11}$$

Taking curl of above equation

$$\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{v}) + \vec{\nabla} \times \left(\vec{v} \times (\vec{\nabla} \times \vec{v})\right) = 0 \tag{1.12}$$

This is a form of Euler equation only in terms of velocity.

Numerous phenomena can be explained using Continuity equation and Euler equation. Most of the fluid dynamics problems happen to be special cases of these equations. In fact,a lot of phenomena can be explained by equations which are special cases of these equations. In the following sections we discuss applications of these equations in different cases.

1.4 Hydrostatics

It is a branch of Hydro dynamics that deals fuilds in mechanical equillibrium. Although the word “Hydrodynamics” is of greek origin which means “water at rest" the study of Hydrodynamics encompasses all fluids in mechanical equillibrium.

A fluid is said to be in mechanical equillibrium if each fluid element experiences zero unbalanced force. Hence all the fluid elements move with a constant velocity in inertial frames where net force is zero. We can choose an inertial frame in which the velocity of fluid is zero. In this frame we have

$$\vec{v}(x, y, z, t) = 0$$

Putting $\vec{v} = 0$ in the Euler equation we obtain In the above derivation ρ is assumed to be constant. Such is the case of fluids for short distances. This is effective in systems like ponds, pools of fluid like swimming pools, tank containing liquids etc. which are encountered in many practical problems. In general, ρ need not be constant. One of the important example is that of gases.

$$\frac{1}{\rho}\vec{\nabla} p = \vec{g} \tag{1.13}$$

Let us choose a coordinate system in which $\vec{g} = g\hat{z}$. Then we have

$$\frac{\partial p}{\partial x}\hat{x} + \frac{\partial p}{\partial y}\hat{y} + \frac{\partial p}{\partial z}\hat{z} = \rho g\hat{z} \tag{1.14}$$

This implies that

$$\frac{\partial p}{\partial x} = 0$$
$$\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial p}{\partial z} = -\rho g$$

The solution of above differential equations is

$$p = -\rho g z + c \tag{1.15}$$

where c is a constant. Suppose at height h ($z = h$) pressure is p_0 then $p_0 = -\rho g h + c$. Pyutting the value of c from this equation we obtain

$$p = p_0 + \rho g(h - z) \tag{1.16}$$

This is the case when the system is in mechanical equillibrium. Suppose it is also in thermal equillibrium, then the temperature is same at every point. We have thermodynamic equation

$$d\phi = -s dT + \frac{1}{\rho} dp \tag{1.17}$$

where ϕ is the Gibbs free energy per unit volume. For constant temperature we have $d\phi = dp/\rho$, which in three dimensions is

$$\overrightarrow{\nabla}\phi = \frac{1}{\rho}\overrightarrow{\nabla}p$$

For hydrostatic problems $\overrightarrow{\nabla}p = \rho \cdot \vec{g}$ Using this we have

$$\overrightarrow{\nabla}\phi = -\vec{g} \tag{1.18}$$

Since $\vec{g} = g\hat{z}$, we have

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= 0 \\ \frac{\partial \phi}{\partial y} &= 0 \\ \frac{\partial \phi}{\partial z} &= -g \end{aligned}$$

The solution of these equations is $\phi = -gz + c$, where c is a constant. Hence the condition of thermal equillibrium in an external field is

$$\phi + gz = c \tag{1.19}$$

Now we have developed suffiecient tools to claim that a horizontal temperature variation disturbs mechanical equillibrium. This can be seen in the hydrostatic problem in a vertical gravitational field. We have

$$\begin{aligned} &\overrightarrow{\nabla}p = -\rho g\hat{z} \\ \Longrightarrow &\frac{\partial \rho}{\partial z} = -\rho g \\ \Longrightarrow &\rho = -\frac{1}{g}\frac{dp}{dg} = \rho(z) \end{aligned}$$

As p and ρ depend on z only, the equation of state requires that $T = T(z)$. If there is temperature variation along x or y, then mechanical equillibrium is not possible.

1.5 Bernoulli Equation

Several practical situations are encountered in which the velocity of a fluid at a particular point remains constant. The velocity of water flowing smoothly in a pipe for a long time and the flow of blood in the blood vessels are some of the important examples. The flow in which the velocity of fluid element is constant in time is known as steady flow. In this case

$$\frac{\partial \vec{v}}{\partial t} = 0$$

So the Euler equation becomes

$$\vec{v} \times (\vec{\nabla} \times \vec{v}) = -\vec{\nabla}\left(w + \frac{1}{2}v^2\right)$$

This implies

$$\vec{\nabla}\left(\frac{1}{2}v^2\right) + \vec{v} \times (\vec{\nabla} \times \vec{v}) = -\vec{\nabla}(w) \tag{1.20}$$

We now define streamlines as one dimensional curves such that tangent to it gives the direction of the velocity at that point. It is defined as a line which is everywhere parallel to the local velocity vector $\vec{v}(x, y, z, t) = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$, $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along x, y and z axes respectively. An infinitesimal diplacement vector in the fluid is

$$\vec{dS} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

To get the equation of streamlines we use the definition and demand that the displacement vector $\vec{dS}$ to be parallel to velocity at a point. Thus

$$\vec{dS} \times \vec{v} = 0$$

In terms of it's $x-$component

$$\begin{aligned} (\vec{dS} \times \vec{v})_x &= 0 \\ \Longrightarrow v_z dy - v_y dx &= 0 \\ \Longrightarrow \frac{dy}{v_y} - \frac{dx}{v_x} &= 0 \end{aligned} \tag{1.21}$$

Similar evaluation for other two components yields the equation of streamlines. The system of equations for streamlines is given by

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} \tag{1.22}$$

For a steady flow, streamlines coincide with the fluid particles. An interesting property of the streamline is that the energy per unit mass remains the same along a streamline. Let the unit vector $\hat{l}$ be the tangent to the streamline. The projection of eqn. 1.20 along $\hat{l}$ is

$$
\begin{aligned}
&\left(\vec{\nabla}\left(w+\frac{v^2}{2}\right)\right)\cdot\hat{l}=0\\
\Longrightarrow\ &\frac{\partial}{\partial l}\left(w+\frac{v^2}{2}\right)=0\\
\Longrightarrow\ &w+\frac{v^2}{2}=c
\end{aligned}
$$

where c is a constant. The fact that $\hat{l}$ is along the direction of velocity $\vec{v}$, renders the second term on the left hand side of eqn. 1.20 to vanish. If the flow takes place in a gravitational field then we have another term

$$\vec{g}\cdot\hat{l}=-g\hat{z}\cdot\vec{l}=-g\vec{\nabla}\cdot\hat{l}=-g\frac{dz}{dl}$$

So, now we have

$$\frac{v^2}{2}+w+gz=c \tag{1.23}$$

where c is a constant. This equation can be interpreted as a manifestation of conservation of energy along a streamline. The first term $v^2/2$ is the kinetic energy per unit mass, the second term w represents the heat function per unit mass and the last term gz is the gravitational potential energy per unit mass. The above equation states that the sum of these forms energy remains the same at each and every point of a streamline. This equation is useful for determination of velocity of a steadily flowing fluid, like water in a pipe etc. Moreover it is also useful for designing the wings of aeroplanes, so that it gets more upthrust during take off.

1.6 Energy Flux

Flowing fluid transports energy from one one point to another. Therefore it is useful to determine the amount of energy flowing per unit cross section area per unit time. This quantity is known as the Energy Flux. We choose a volume element fixed in space. Now we need to find the time dependence of energy contained in this volume. Let ϵ be the internal energy per unit mass. So, energy of unit volume of the fluid is

$$\frac{1}{2}\rho v^2+\rho\epsilon$$

We are interested in explicit rate of change of this quantity with respect to time. Consider the first term, that is, the rate of change of kinetic energy density

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{1}{2}v^2\frac{\partial \rho}{\partial t} + \rho\vec{v}\cdot\frac{\partial \vec{v}}{\partial t}$$

Now using the continuity equation and the Euler equation we have

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\frac{v^2}{2}\vec{\nabla}(\rho\vec{v}) + \vec{v}\left(-\vec{\nabla}p - \rho(\vec{v}\cdot\vec{\nabla})\vec{v}\right) \tag{1.24}$$

From thermodynamics we have $dw = Tds + \frac{1}{\rho}dp$. So,

$$\vec{\nabla}w = T\vec{\nabla}s + \frac{1}{\rho}\vec{\nabla}p$$

After a trivial manipulation we obtain

$$\rho\vec{v}\cdot\vec{\nabla}w - \rho T\vec{v}\cdot\vec{\nabla}s = \vec{v}\cdot\vec{\nabla}p$$

Using this in eqn. 1.24 we get

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\frac{v^2}{2}\vec{\nabla}(\rho\vec{v}) - \rho\vec{v}\cdot\vec{\nabla}\left(\frac{1}{2}v^2 + w\right) + \left(\rho T\vec{v}\cdot\vec{\nabla}s\right)$$

Now consider the rate of change of internal energy density with respect to time

$$\frac{\partial}{\partial t}(\rho\epsilon)$$

Note that

$$d\epsilon = Tds - pdV \implies Tds + \frac{p}{\rho^2}d\rho$$

Here we have used the fact that V is the specific volume, that is, volume per unit mass. Hence $1/V = \rho$. The heat function per unit mass is then $w = \epsilon + p/\rho$. Now

$$\begin{aligned} d(\rho\epsilon) &= \epsilon d\rho + \rho d\epsilon \\ &= \left(w - \frac{p}{\rho}\right)d\rho + \rho\left(Tds + \frac{p}{\rho^2}d\rho\right) \\ &= wd\rho + \rho Tds \end{aligned}$$

Therefore the rate of change of internal energy density becomes

$$\begin{aligned} \frac{\partial}{\partial t}(\rho\epsilon) =& w\frac{\partial \rho}{\partial t} + \rho T\frac{\partial s}{\partial t} \\ =& -w\vec{\nabla}(\rho\vec{v}) - \rho T\vec{v}\cdot\vec{\nabla}s \end{aligned} \tag{1.25}$$

Using eqn. 1.24 and 1.25, the time derivative of energy density is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\epsilon\right) = -\overrightarrow{\nabla}\left(\rho\vec{v}\left(w + \frac{v^2}{2}\right)\right) \tag{1.26}$$

Integrating this equation over a volume large enough such that it's surface has no fluid ($\rho = 0$)

$$\frac{\partial}{\partial t}\int_V dV\left(\frac{1}{2}\rho v^2 + \rho\epsilon\right) = -\int_V \overrightarrow{\nabla}\left(\rho\vec{v}\left(w + \frac{v^2}{2}\right)\right) \tag{1.27}$$

Using Gauss law

$$\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\rho v^2 + \rho\epsilon\right) = -\oint_S \left(\rho\vec{v}\left(w + \frac{v^2}{2}\right)\right)\hat{n}dS \tag{1.28}$$

Here S is the surface enclosing the volume V. The integrand on the RHS is the energy per unit area per unit time flowing out of the volume V. Hence, it is also known as energy flux density vector.

To understand the physical significance of energy flux density we use the relation $w = \epsilon + p/\rho$. Now the right hand side can be written as

$$-\oint_S \overrightarrow{\nabla}\left(\rho\vec{v}\left(\epsilon + \frac{v^2}{2} + p/\rho\right)\right)\hat{n}dS$$

The first two terms represent the internal energy and kinetic energy flowing out of the volume. And the term with a factor p/ρ in the integrand is the work done by the forces which is manifested as pressure on the system.

1.7 Momentum Flux

Flowing fluids also carry momentum with them. To determine the flow of momentum we choose a particular volume in the fluid and see the time rate of change of momentum in it. Here, and in further sections, we shall work with the components of vectors. The i^{th} component of momentum density at a point (x, y, z) is $\rho(x, y, z)v_i(x, y, z)$. For repeated indices, we shall use Einstein's summation convention. Also for the derivatives we shall use the convention

$$\frac{\partial}{\partial t} \rightarrow \partial_t$$

$$\text{and } \frac{\partial}{\partial x_k} \rightarrow \partial_k$$

The time derivative of momentum density is

$$\frac{\partial}{\partial t}(\rho v_i) = \rho \frac{\partial v_i}{\partial t} + \frac{\partial \rho}{\partial t} v_i \tag{1.29}$$

Using equation of continuity

$$\frac{\partial v_i}{\partial t} = -v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i}$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho v_i) &= -\rho v_k \frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_k} - v_i \frac{\partial}{\partial x_k}(\rho v_k) \\
&= \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_k}(\rho v_i v_k) \\
&= \delta_{ik} \frac{\partial p}{\partial x_k} - \frac{\partial}{\partial x_k}(\rho v_i v_k) \\
&= \frac{\partial}{\partial x_k}(\delta_{ik} p + \rho v_i v_k)
\end{aligned}$$

Let's define

$$\Pi_{ik} = \delta_{ik} p + \rho v_i v_k \tag{1.30}$$

Note that Π_{ik} is a symmetric tensor, that is, $\Pi_{ik} = \Pi_{ki}$. The rate of change of energy density becomes

$$\partial_t(\rho v_i) = -\partial_k \Pi_{ik} \tag{1.31}$$

Integrating the equation over a volume

$$\frac{\partial}{\partial t}\int (\rho v_i) dV = -\int \partial_k \Pi_{ik} dV = -\oint \Pi_{ik} \widehat{n_k} dS \tag{1.32}$$

Gauss law has been used in the last step. Here $\widehat{n_k}$ is the k^{th} component of unit normal of the surface S. The left hand side of the equation is the rate of change of the i^{th} component of the momentum contained in the volume. The surface integral on the right is the amount of momentum flowing out through the surface bounding the volume V per unit time. The integrand of the surface integral $\Pi_{ik}\widehat{n_k}$ is the flux of i^{th} component of momentum through unit surface area. Hence Π_{ik} is the i^{th} component of the amount of momentum flowing per unit time through a unit area perpendicular to x_k-axis. So Π_{ik} is also called Momentum flux density tensor. Note that a physical quantity of rank r has it's flux density of rank $r + 1$.

Let us observe the component of Π_{ik}, parallel and perpendicular to the direction of velocity $\vec{v}$. We have

$$\Pi_{ik}\hat{n_k} = \delta_{ik}\hat{n_k} + \rho v_i v_k \hat{n_k} \tag{1.33}$$

If $\hat{n_k}$ is along the direction of velocity then $\rho v_k v_i \hat{n_k}$ is $\rho v^2 \hat{n_i}$. And $\Pi_{ik}\hat{n_k} = (p + \rho v^2) n_i$. This is the longitudinal component of momentum. Onthe other hand if $\hat{n_k}$ is perpendicular to the direction of velocity the $v_k \hat{n_k} = 0$, which gives $\Pi_{ik}\hat{n_k} = p\hat{n_i}$. This is the transverse component of momentum.

1.8 Conservation of Circulation and Potential Flow

Consider a closed contour C in the fluid. The integral

$$\Gamma = \oint_C \vec{v} \cdot \vec{dl}$$

is called the velocity circulation round that contour. Consider a closed drawn in the fluid at some instant. It is composed of fluid particles lying on it. The time derivative of the circulation is

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \oint \vec{v} \cdot \vec{dl} \tag{1.34}$$

Here we have taken total time derivative because we want change in circulation round a "fluid contour" as it moves about. For a fluid contour fixed in space we would have calculated partial derivative with respect to time. In the following part of the section we use the notation δ for space derivative and d for time derivative. So the time derivative of circulation is

$$\begin{aligned}\frac{d\Gamma}{dt} =& \frac{d}{dt} \oint \vec{v} \cdot \delta\vec{r} \\ =& \oint \frac{d\vec{v}}{dt} \delta\vec{r} + \oint \vec{v} \frac{d}{dt}(\delta\vec{r})\end{aligned}$$

Since $\vec{v} = d\vec{r}/dt$, we have

$$\vec{v}\frac{d}{dt}(\delta\vec{r}) = \vec{v}\delta\frac{d\vec{r}}{dt} = \delta(v^2/2)$$

This gives an exact differential and hence it's integral vanishes. It is easy to see from Euler equation that

$$\frac{d\vec{v}}{dt} = -\vec{\nabla} w$$

Using this we obtain

$$\begin{aligned}\oint \frac{d\vec{v}}{dt}\delta\vec{r} &= \int_S (\overrightarrow{\nabla} \times \frac{d\vec{v}}{dt})\delta(\hat{n}dS) \\ &= \int_S (\overrightarrow{\nabla} \times (-\overrightarrow{\nabla} w))\delta(\hat{n}dS) \\ &= 0 \\ \implies \Gamma &= \oint \vec{v} \cdot \vec{dl} = c\end{aligned}$$

where c is a constant. Using Stokes law we conclude that

$$\oint \vec{v} \cdot \vec{dl} = \delta\vec{S} \cdot \overrightarrow{\nabla} \times v = c$$

The last step follows from the fact that the curl of a gradient vanishes. Thus the circulation Γ is a constant. Hence we proved that in an ideal fluid the velocity circulation round a closed fluid contour is constant in time. This is also known as Kevin's theorem or law of conservation of circulation.

Conservation of circulation leads to important consequences. Lets us first consider that the flow is steady. Consider a streamline which has a point satisfying $\overrightarrow{\nabla} \times \vec{v} = 0$. So the line integral of fluid velocity over an infinitesimal small circular loop around this point vanishes. This loop will always encircle the streamline. Since we have Γ to be 0, which is a constant for this loop, we can always assure that $\overrightarrow{\nabla} \times \vec{v} = 0$ for all points of the streamline. This empowers us to argue as follows. Consider steady flow past some body. If the incident flow is uniform, so that it has a vanishing curl of velocity, we shall always have curl of velocity equal to zero. SUch a flow for which curl of velocity vanishes everywhere is known as potential flow or irrotational flow. The name "Irrotational" is self evident because of vanishing curl. It is also called potential flow because the velocity, having a vanishing curl, can be expressed as a gradient of a scalar potential.

Now let us suppose that the at some instant we have potential flow throughout the volume of the fluid. The velocity circulation Γ for any arbitrary closed loop will vanish. According to Kevin's theorem we have that this will be constant in time. Hence the curl of velocity will be zero at any future time also.

As discussed earlier, for potential flows we can define a scalar function ϕ called the velocity potential.

$$\vec{v} = \overrightarrow{\nabla}\phi \tag{1.35}$$

Consider the Euler equation in the following form

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{2}\vec{\nabla} v^2 - \vec{v} \times (\vec{\nabla} \times \vec{v}) = -\vec{\nabla} w \tag{1.36}$$

Using eqn. 1.35 in above equation we obtain

$$\begin{aligned} \vec{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{v^2}{2} + w\right) &= 0 \\ \implies \frac{\partial \phi}{\partial t} + \frac{v^2}{2} + w &= f(t) \end{aligned} \tag{1.37}$$

where $f(t)$ is an arbitraty function of time. We can impose $f(t) = 0$ as ϕ is not unique ($\vec{v} = \vec{\nabla}(\phi + f(t)) = \vec{\nabla}\phi$). For a steady flow we have

$$\frac{\partial \phi}{\partial t} = 0$$

and

$$f(t) = \text{constant}$$

So we have

$$\frac{v^2}{2} + w = \text{constant} \tag{1.38}$$

Note that it is of the same form as Bernoulli equation. But it is a more powerful condition as in this case the constant is for the entire fluid. Unlike Bermoulli equation in which constant is for a given streamline, the constant in above equation is the same for all stramlines of the fluid.

1.9 Incompressible Fluids

If ρ is constant we say that the fluid is incompressible. By compressibility we mean the change in volume of a given mass by some process. Negative compressibility is defined as decrease in volume of a given mass. On the other hand we define positive compressibility as increase in volume of a given mass. Change in volume of a given mass changes the density. So by fixing the density ρ we get the condition for incompressibility. In this case the Euler equation can be written as

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\vec{\nabla})\vec{v} = -\vec{\nabla}\left(\frac{p}{\rho}\right) + \vec{g}$$

The equation of continuity, in case of steady flow reduces to the fact that the velocity field becomes divergenceless, that is

$$\vec{\nabla} \cdot \vec{v} = 0$$

Enthalpy per unit mass becomes $w = p/\rho$ and hence we have

$$\frac{v^2}{2} + \frac{p}{\rho} + gz = \text{constant}$$

The energy flux in this case becomes

$$\rho\vec{v}\left(\frac{v^2}{2} + \frac{p}{\rho} = 0\right)$$

For incompressible fluids exhibiting isentropic flow there will be no change in internal energy per unit mass. This is evident from the definition that

$$d\epsilon = Tds - pd\left(\frac{1}{\rho}\right)$$

The condition for potential flow $\vec{v} = \vec{\nabla}\phi$ and incompressibility $\vec{\nabla}\vec{v}$ leads to Laplace equation

$$\nabla^2\phi = 0 \tag{1.39}$$

If appropriate boundary conditions are supplied then the velocity field can be uniquely determined. The two boundary condition is a constraint on the normal component of fluid velocity relative to the boundary surface. For fixed surfaces, fluid velocity component v_n normal to the surface must vanish. For moving surface it is equal to normal component of moving body.

Chapter 2

Non-Relativistic Viscous Fluid

Ideal fluids do not dissipate energy along the course of it's flow. To describe real fluids it is necessary to take into consideration the effects of energy losses through dissipation. By work energy theorem we know that the total change in kinetic energy of a mechanical system is the total amount of work done by all the forces on it. In our analysis we shall consider mechanical energy dissipation, which, we expect to be manifested as decrease in kinetic energy density of the fluid. In this chapter, we introduce a term that would be responsible for decrease in kinetic energy of the system with time. Also we shall give physical interpretations to the terms introduced for dissipation.

2.1 Viscous Stress Tensor

The Euler equation without any dissipation is

$$\partial_t(\rho v_i) = \partial_k \Pi_{ik}$$

As discussed in previous chapter, this Π_{ik} is a component of Momentum flux density tensor (i^{th} component of momentum flowing through the surface wirh a normal along x_k axis). We have

$$\Pi_{ik} = \delta_{ik} p + \rho v_i v_k$$

Now we introduce another term denoted by σ'_{ik} such that the momentum flux density tensor becomes

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \sigma'_{ik} \tag{2.1}$$

The term σ'_{ik} has a factor of -1 which indicates lowering of momentum flux density as compared to the flux density tensor for ideal fluid. Now let us define

$$\sigma_{ik} = \sigma'_{ik} - p\delta_{ik} \tag{2.2}$$

The term σ_{ik} gives the part of momentum flux that is not due to direct transfer of momentum with the mass of moving fluid. The quantity σ_{ik} is known as viscous stress tensor. Thus the momentum flux density tensor becomes

$$\Pi_{ik} = \rho v_i v_k - \sigma_{ik} \tag{2.3}$$

We shall now try to make an educated guess of the form of σ'_{ik}. Internal friction processes occur only if the fluid particles move with a relative velocity with respect to each other. This implies that there is a relative velocity between different layers of fluid and hence σ'_{ik} depends on $\partial_k v_i$ for small velocity gradients. Linear dependence with velocity gradient can be assumed till first approximation. For better approximations we need to consider dependences on higher powers of $\partial_k v_i$. But we shall assume linear dependence henceforth in the discussion. We require that for constant velocity there should be no dissipation and hence σ'_{ik} must vanish. This is satisfied only if all the terms in σ'_{ik} all the terms in σ'_{ik} are velocity gradients. Also in order to satisfy rotational equillibrium, we need σ'_{ik} to vanish for uniform rotation.This leads to the fact that if there is a gradient of i^{th} component of velocity along $\hat{n_k}$ direction, then we must have a gradient of k^{th} component of velocity along $\hat{n_k}$. This is the condition of symmetry and requires σ'_{ik} to have a dependence on $\partial_i v_k + \partial_k v_i$ upto some constant (say η).

In order to prove this let us consider a constant angular velocity $\overrightarrow{\Omega}$. The velocity $\vec{v}$ is given by

$$\vec{v} = \overrightarrow{\Omega} \times \vec{r} \tag{2.4}$$

In this case we have

$$\begin{aligned}
\partial_k v_i + \partial_i v_k &= \partial_k(\overrightarrow{\Omega} \times \vec{r})_i + \partial_i(\overrightarrow{\Omega} \times \vec{r})_k \\
&= \partial_k \epsilon_{ilm} \Omega_l r_m + \partial_i \epsilon_{klm} \Omega_l r_m \\
&= \epsilon_{ilm} \Omega_l \delta_{km} + \epsilon_{klm} \Omega_l \delta_{im} \\
&= \epsilon_{ilk} \Omega_l + \epsilon_{kli} \Omega_l \\
&= (\epsilon_{ilk} + \epsilon_{kli}) \Omega_l \\
&= (\epsilon_{ilk} - \epsilon_{ilk}) \Omega_l \\
&= 0
\end{aligned}$$

Again we need the viscous stress to have a traceless term for shear viscosity and a term with non-vanishing trace for bulk viscosity. The most general form of σ'_{ik}, satisfying above requirements is

$$\sigma'_{ik} = \eta\left(\partial_k v_i + \partial_i v_k - \frac{2}{3}\delta_{ik}\partial_l v_l\right) + \xi \delta_{ik} \partial_l v_l \tag{2.5}$$

To verify tracelessness of the first term we simply contract an index in the above expression

$$\eta\left(2\partial_k v_k - \frac{2}{3}\delta_{kk}\partial_l v_l\right) = 0$$

In the last step we have used the trace of an identity matrix to be $\delta_{kk} = 3$. In the expression of σ'_{ik}, η and ξ are positive and are independent of velocity.

Equation of viscous fluid now can be derived by adding $\partial_k\sigma'_{ik}$ to the Euler Equation of Motion. Viscous effects when added we obtain

$$\rho\left(\partial_t v_i + v_k\partial_k v_i\right) = -\partial_i p - \partial_k\sigma'_{ik} \tag{2.6}$$

The equation of motion in terms of fluid state variables is

$$\rho\left(\partial_t v_i + v_k\partial_k v_i\right) = -\partial_i p + \partial_k\left(\eta(\partial_k v_i + \partial_i v_k - \frac{2}{3}\delta_{ik}\partial_l v_l) + \xi\partial_i(\partial_l v_l)\right) \tag{2.7}$$

The above equation written in terms of vector instead of components of vectors we obtain

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right) = -\vec{\nabla}p + \eta\nabla^2\vec{v} + \left(\frac{\eta}{3} + \xi\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) \tag{2.8}$$

This is the general form of Navier-Stokes Equation. This equation can be modified for various situations such as incompressibility, potential flow, etc.

For an incompressible fluid ($\vec{\nabla} = 0$), the Navier Stokes equations becomes

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho} + \frac{\eta}{\rho}\nabla^2\vec{v} \tag{2.9}$$

The stress tensor for incompressible fluid is

$$\begin{aligned}\sigma_{ik} =& \sigma'_{ik} - p\delta_{ik}\\ =& \eta\left(\partial_k v_i + \partial_i v_k\right) - p\delta_{ik}\end{aligned} \tag{2.10}$$

The viscosity of an incompressible fluid is determined by only one component η, that is, the shear component of viscosity.

2.2 Equation of Motion for Viscous Fluid

Force acting on a surface element is equal to the momentum flux through the surface element. Consider a surface element $\vec{da}$. The i^{th} component infinitesimal force on this surface is

$$dF_i = \Pi_{ik}n_k da \tag{2.11}$$

Now we shall use $\Pi_{ik} = \rho v_i v_k - \sigma_{ik}$ and the fact boundary condition that the velocity of fluid at boundary vanishes. Therefore

$$\begin{aligned} dF_i &= -\sigma_{ik} da \\ &= \left(-\sigma'_{ik} + p\delta_{ik}\right) da \end{aligned}$$

We have two terms in the expression of force element

$$dF_i = pn_i da - \sigma'_{ik} n_k da \tag{2.12}$$

The first term is the ordinary pressure on the area element da. And the second term is the force of friction. The negative sign ensures that it opposes the motion.

We require to to match the boundary conditions to get the continuity of velocity field of the fluid. When the boundary is a fixed solid surface we require the velocity (both normal and tangential components) to be zero.

For two immiscible fluids we require

$$\sigma_{1,ik} n_{1,k} + \sigma_{2,ik} n_{2,k} = 0 \tag{2.13}$$

For a given interface we have $\hat{n}_{1,k} = -\hat{n}_{2,k} = \hat{n}$, then we have

$$\sigma_{1,ik} n_i = \sigma_{2,ik} n_i$$

The equation of motion is given by the Euler equation of motion with a term of space derivative of viscous stress tensor added it. For viscous fluids the equation of motion is given by

$$\rho \left(\partial_t v_i + v_k \partial_k v_i\right) = -\partial_i p + \partial_k \sigma_{ik} \tag{2.14}$$

where

$$\sigma_{ik} = \eta \left(\partial_k v_i + \partial_i v_k - \frac{2}{3}\delta_{ik}\partial_l v_l\right) + \xi \delta_{ik} \partial_l v_l$$

2.3 Energy Dissipation in incompressible fluid

In this section we intend to prove that the kinetic energy density of the fluid decreases with time. For simplicity, we consider only the incompressible fluid ($\overrightarrow{\nabla} \vec{v} = 0$). The kinetic energy of an incompressible fluid is

$$E_{kin} = \frac{1}{2}\rho \int v^2 dV \tag{2.15}$$

From eqn. 2.14 we obtain

$$\frac{\partial v_i}{\partial t} = -v_k \partial_k v_i - \frac{1}{\rho}\partial_i p + \frac{1}{\rho}\partial_k \sigma'_{ik} \tag{2.16}$$

In this equation we use

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \vec{v}\rho\frac{\partial \vec{v}}{\partial t} = v_i \rho \frac{\partial v_i}{\partial t}$$

and then multiplying with ρ we obtain,

$$\rho v_i \partial_t v_i = -\rho v_i v_k \partial_k v_i - v_i \partial_i p + v_i \partial_k \sigma'_{ik} = 0 \tag{2.17}$$

This equation when written in terms of vectors

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\overrightarrow{\nabla}\left[\rho\vec{v}\left(p + \frac{\rho v^2}{2}\right) - \vec{v}\vec{\sigma}'\right] - \sigma'_{ik}\partial_k v_i \tag{2.18}$$

Note that the first term inside the divergence is the energy flux density of the the ideal fluid. The term $\vec{v}\vec{\sigma}'$ is the flux density due to internal friction.

Integrating eqn. 2.18 over a large volume V, we obtain

$$\int \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) dV = -\int \overrightarrow{\nabla}\left[\rho\vec{v}\left(\frac{1}{2}\rho v^2 + \frac{p}{\rho}\right) - \vec{v}\cdot\vec{\sigma}'\right] dV - \int \sigma'_{ik}\frac{\partial c_i}{\partial x_k} dV \tag{2.19}$$

The first term can be converted to a surface integral by Gauss Law. This integral vanishes for localised fluid. So we are left with

$$\int \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) dV = -\frac{1}{2}\int \sigma'_{ik}\frac{\partial v_i}{\partial x_k} dV \tag{2.20}$$

The right hand side of above equation is the rate of decrease of kinetic energy owing to dissipation. Using symmetry of σ'_{ik} we can write

$$\int \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) dV = -\frac{1}{2}\int \sigma'_{ik}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) dV \tag{2.21}$$

Using the expression $\sigma'_{ik} = \eta(\partial_k v_i + \partial_i v_k)$ for incompressible fluids we obtain

$$\frac{dE_{kin}}{dt} = -\frac{1}{2}\eta \int \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)^2 dV \tag{2.22}$$

Since dE_{kin}/dt is negative of a positive definite quantity, the kinetic energy always decreases with time. Hence mechanical dissipation has been accounted by the introduction of the stress tensor term.

Chapter 3

Relativistic Hydrodynamics.

The Euler equation is basically a manifestation of Newton's second law which is not relativistic, that is, not invariant under lorentz transformation. Therefore it works well for small velocities as compared to the speed of light. In the limit $v \to c$ we need to consider relativistic equations.

For a non-relativistic system the degrees of freedom were the fluid density, pressure, and the three components of velocity ρ, p, v_x, v_y, v_z. But ρ does not account for kinetic energy contributions of terms higher order in v/c, which becomes significant as v/c tends to unity. Since mass is equivalent to energy, let's replace the mass density ρ by the energy density ϵ. Similarly, we replace the velocity $\vec{v}(\vec{x}, t)$ by the four velocity

$$u^\mu = \frac{dx^\mu}{dt}$$

where τ is the proper time. Here the convention used for $g^{\mu\nu}$ is $(+, -, -, -)$.

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - \vec{dx} \cdot \vec{dx} \tag{3.1}$$

Here we use the natural units where $c = 1$. An interval of time is not an absolute quantity, but a relative one. If the time interval is $d\tau$ in a frest frame then the interval in a frame moving with velocity v with respect to the rest frame is given by

$$dt = \gamma d\tau \tag{3.2}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}}$$

The four velocity is given by

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma(1, \vec{v}) \tag{3.3}$$

To obtain relativistic fluid dynamic equations, we need to derive the energy momentum tensor $T^{\mu\nu}$ for a relativistic fluid.

3.1 Energy Momentum Tensor

The energy momentum tensor $T_0^{\mu\nu}$ has to be built out of hydrodynamic degrees of freedom, viz. lorentz scalars ϵ, p, and and u^μ and $g^{\mu\nu}$. The subscript of $T_0^{\mu\nu}$ denotes that it is the energy-momentum tensor for ideal fluid. Also the energy momentum tensor has to be symmetric. The most general form of Energy Momentum Tensor satisfying these conditions is

$$T_0^{\mu\nu} = \epsilon(c_0 g^{\mu\nu} + c_1 u^\mu u^\nu) + p(c_2 g^{\mu\nu} + c_3 u^\mu u^\nu) \tag{3.4}$$

In the local rest frame we require $T_0^{\mu\nu}$ to have following properties

1. T_0^{00} represents energy density ϵ of the fluid.
2. Momentum density should vanish, i.e. $T_0^{0i} = 0$.
3. Space-like components should be proportional to pressure $T_0^{ij} = p\delta_{ij}$

Applying condition (1), (2) and (3) we get

$$\epsilon = (c_0 + c_1)\epsilon + (c_2 + c_3)p$$

and

$$p = -c_0\epsilon - c_2 p$$

Solving these equations gives $c_0 = 0, c_1 = c_3 = 1$ and $c_2 = -1$. Using these values of c_i $(i = 0, 1, 2, 3)$, we obtain

$$T_0^{\mu\nu} = \epsilon u^\mu u^\nu - p(g^{\mu\nu} - u^\mu u^\nu) \tag{3.5}$$

Let us introduce the tensor $\Delta^{\mu\nu}$ defined by

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \tag{3.6}$$

The tensor $\Delta^{\mu\nu}$ has following properties.

1. $\Delta^{\mu\nu} u_\mu = (g^{\mu\nu} - u^\mu u^\nu) u_\mu = u^\nu - u^\nu = 0$
2. $\Delta^{\mu\nu} \Delta_\nu^\alpha = \Delta^{\mu\alpha}$

This serves as a projection operator on the space orthogonal to the fluid velocity u^μ. The energy momentum tensor of an ideal relativistic fluid, in this notation becomes

$$T_0^{\mu\nu} = \epsilon u^\mu u^\nu - p\Delta^{\mu\nu} \tag{3.7}$$

3.2 Relativistic Ideal Fluids

We shall show in this section that the fluid dynamics equations are a manifestation of conservation equations for fluid's energy momentum tensor. If there are no external sources the energy momentum tensor is conserved. So we have

$$\partial_\mu T_0^{\mu\nu} = 0 \tag{3.8}$$

Projecting eqn. 3.8 parallel to the fluid velocity we obtain

$$u_\nu \partial_\mu T_0^{\mu\nu} = u_\nu \partial_\mu (\epsilon u^\mu u^\nu - p\Delta^{\mu\nu})$$

Using the fact that $u^\mu u_\mu = 1$ and hence $u_\mu \partial_\nu u^\nu = 0$, and simple algebraic manipulation leads to

$$u_\nu \partial_\mu T_0^{\mu\nu} = u^\mu \partial_\mu \epsilon + \epsilon(\partial_\mu u^\mu) + \epsilon u_\nu u^\mu \partial_\mu u^\nu - p u_\mu \partial_\nu \Delta^{\mu\nu} \tag{3.9}$$

This implies

$$u_\nu \partial_\mu T_0^{\mu\nu} = (\epsilon + p)\partial_\mu u^\mu + u^\mu \partial_\mu \epsilon \tag{3.10}$$

The other projection gives

$$\Delta_\nu^\alpha \partial_\mu T_0^{\mu\nu} = \epsilon u^\mu \Delta_\nu^\alpha \partial_\mu u^\nu - \Delta^{\mu\alpha}(\partial_\mu p) + p u^\mu \Delta_\nu^\alpha \partial_\mu u^\nu \tag{3.11}$$

Arranging the terms in above equation leads to

$$\Delta_\nu^\alpha \partial_\mu T_0^{\mu\nu} = (\epsilon + p) u^\mu \partial_\mu u^\alpha - \Delta^{\mu\alpha} \partial_\mu p \tag{3.12}$$

Introducing short hand notations

$$D = u^\mu \partial_\mu \tag{3.13}$$

and

$$\nabla^\alpha = \Delta^{\mu\alpha} \partial_\mu \tag{3.14}$$

Using above notations in the equation of perpendicular (eqn. 3.12) and parallel (eqn. 3.10) projection, eqn. 3.8 is decomposed to following two equations

$$D\epsilon + (\epsilon + p)\partial_\mu u^\mu = 0 \tag{3.15}$$

and

$$(\epsilon + p) D u^\alpha - \nabla^\alpha p = 0 \tag{3.16}$$

Equations 3.15 and 3.16 are the fundamental equations of relativistic ideal fluid. It is evident that the equations are relativistic invariant. In order to verify these equations we take the non-relativistic limit i.e. $v << 1$ (in the natural units). In the non-relativistic limit we have

$$u^\mu = \gamma(1, \vec{v}) \approx (1, \vec{0}) \tag{3.17}$$

In this limit we have

$$\begin{aligned} D &= u^\mu \partial_\mu \\ &\approx \partial_t + \vec{v} \cdot \vec{\partial} + O(v^2) \end{aligned}$$

So eqn. 3.15 becomes

$$\begin{aligned} &D\epsilon + (\epsilon + p)\partial_\mu u^\mu = 0 \\ \Longrightarrow\ &\partial_t \epsilon + \vec{v} \cdot \vec{\partial}\epsilon + \epsilon \vec{\partial} \cdot \vec{v} = 0 \end{aligned}$$

Now using the fact that in the non-relativistic limit the energy density equals the mass density and the momentum is very small, we have

$$\begin{aligned} &\partial_t \epsilon + \vec{\partial} \cdot (\vec{v}\epsilon) = 0 \\ \Longrightarrow\ &\partial_t \rho + \vec{\partial} \cdot (\vec{v}\rho) = 0 \end{aligned}$$

Hence we obtained the continuity equation as a non-relativistic limiting case of relativistic equation.

Again, in the limit $v << 1$, we have

$$\begin{aligned} \nabla^\alpha &= \Delta^{\mu\alpha}\partial_\mu \\ &= (g^{\mu\nu} - u^\mu u^\nu)\partial_\mu \\ &= \partial^\alpha + O(|\vec{v}|) \end{aligned}$$

This implies

$$\nabla^i = \partial^i \tag{3.18}$$

Therefore eqn. 3.16 becomes

$$\begin{aligned} &(\epsilon + p)(\partial_t + v_k v_k)u^\alpha - \partial^\alpha p = 0 \\ &(\epsilon + p)(\partial_t + v_k \partial_k)v^i = -\partial_i p \\ &\rho(\partial_t + v_k \partial_k)v^i = -\partial_i p \end{aligned}$$

where we have used the fact that $(\epsilon + p) \approx \rho$. This is because at non-relativistic limits $p << 1$ and $\epsilon \approx \rho$ On simple rearrangement it becomes

$$(\partial_t + v_k \partial_k)v^i = -\frac{1}{\rho}\partial_i p \tag{3.19}$$

This is the Euler equation of motion. Hence we derived the Euler equation as the non-relativistic limit of relativistic equations for Ideal Fluid. We thus recognise that the fluid dynamic equations to be identical to the conservation equations for the fluid's energy momentum tensor.

3.3 Relativistic Viscous Hydrodynamics

All the dissipative terms were neglected in the case of ideal fluid. We have to go beyond the ideal fluid limit and include viscosity effects. The energy momentum tensor will now have an additional term to account for the dissipation effects. So, we write the energy momentum tensor as

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu} \tag{3.20}$$

where $T_0^{\mu\nu}$ is the energy-momentum tensor for ideal fluid and $\Pi^{\mu\nu}$ includes the effects of dissipation. For a system without conserved charges, the momentum is due to the flow of energy density. That is

$$u_\mu T^{\mu\nu} = \epsilon u^\nu \tag{3.21}$$

This implies

$$\begin{aligned} & u_\mu T^{\mu\nu} = \epsilon u^\nu \\ \implies & u_\mu(\epsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \Pi^{\mu\nu}) = \epsilon u^\nu \\ \implies & \epsilon u^\nu + u_\mu \Pi^{\mu\nu} = \epsilon u^\nu \end{aligned}$$

This implies

$$u_\mu \Pi^{\mu\nu} = 0 \tag{3.22}$$

We shall again use the conservation of energy-momentum tensor

$$\partial_\mu T^{\mu\nu} = 0 \tag{3.23}$$

Projecting above equation parallel to the direction of fluid velocity we obtain

$$\begin{aligned} u_\nu \partial_\mu T^{\mu\nu} &= u_\nu \partial_\mu (T_0^{\mu\nu} + \Pi^{\mu\nu}) \\ &= D\epsilon + (\epsilon + p)\partial_\mu u^\mu + u_\nu \partial_\mu \Pi^{\mu\nu} \end{aligned}$$

and taking an orthogonal projection we obtain

$$\Delta^\alpha_\nu \partial_\mu T^{\mu\nu} = (\epsilon + p) D u^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu}$$

Therefore the conservation of energy momentum tensor implies

$$\begin{aligned} D\epsilon + (\epsilon + p)\partial_\mu u^\mu + u_\nu \partial_\mu \Pi^{\mu\nu} &= 0 \\ (\epsilon + p) D u^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu} &= 0 \end{aligned} \tag{3.24}$$

In order to simplify the first equation further we shall make use of the identity

$$\partial_\mu = u_\mu D + \nabla_\mu \tag{3.25}$$

This identity can be easily proved if we evaluate it's RHS using the definitions of D and ∇_μ

$$\begin{aligned}
u_\mu D + \nabla_\mu &= u_\mu(u^\nu \partial_\nu) + g_{\mu\alpha}\nabla^\alpha \\
&= u_\mu(u^\nu \partial_\nu) + g_{\mu\alpha}(\Delta^{\nu\alpha}\partial_\nu) \\
&= u_\mu(u^\nu \partial_\nu) + g_{\mu\alpha}(g^{\nu\alpha} - u^\nu u^\alpha)\partial_\nu \\
&= g_{\mu\alpha}g^{\nu\alpha}\partial_\nu \\
&= \delta^\nu_\mu \partial_\nu \\
&= \partial_\mu
\end{aligned}$$

Here we introduce the notation for symmetrization

$$A_{(\mu}B_{\nu)} = \frac{1}{2}(A_\mu B_\nu + B_\nu A_\mu) \tag{3.26}$$

Using $u_\nu \Pi^{\mu\nu} = 0$, we have

$$u_\nu \partial_\mu \Pi^{\mu\nu} = \partial_\mu(u_\nu \Pi^{\mu\nu}) - \Pi^{\mu\nu}\partial_{(\mu}u_{\nu)} = -\Pi^{\mu\nu}\partial_{(\mu}u_{\nu)} \tag{3.27}$$

Using these equations we can rewrite eqn. 3.24 as

$$\begin{aligned}
D\epsilon + (\epsilon + p)\partial_\mu u^\mu - \Pi^{\mu\nu}\partial_{(\mu}u_{\nu)} &= 0 \\
(\epsilon + p)Du^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu} &= 0
\end{aligned} \tag{3.28}$$

The equations in above form are symmetric with respect to indices.

Till now we have not yet specified the form of $\Pi^{\mu\nu}$. In fact, different forms of $\Pi_{\mu\nu}$ will give different theories of relativistic hydrodynamics. Our approach to specify the form of $\Pi^{\mu\nu}$ is to compare with the results of thermodynamics. According to the second law of thermodynamics we require the entropy to be increasing. The relation between entropy density, pressure and temperature is given by

$$\begin{aligned}
\epsilon + p &= Ts \\
Tds &= d\epsilon
\end{aligned} \tag{3.29}$$

The second law of thermodynamics in the covariant form is given by

$$\partial_\mu s^\mu \geq 0 \tag{3.30}$$

where $s^\mu = su^\mu$. Using this we have

$$\begin{aligned} \partial_\mu(su^\mu) &= u^\mu \partial_\mu s + s\partial_\mu u^\mu \\ &= Ds + s\partial_\mu u^\mu \\ &= D\left(\frac{\epsilon + p}{T}\right) + \left(\frac{\epsilon + p}{T}\right)\partial_\mu u^\mu \\ &= \frac{1}{T}(D\epsilon + (\epsilon + p)\partial_\mu u^\mu) \\ &= \frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} \end{aligned} \tag{3.31}$$

Therefore the condition of increasing entropy becomes

$$\frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} \geq 0 \tag{3.32}$$

We now split $\Pi^{\mu\nu}$ into a traceless term and another with a non-vanishing trace as follows.

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Delta^{\mu\nu}\Pi \tag{3.33}$$

We introduce the notation for the traceless part of $\nabla_{(\mu}u_{\nu)}$ as follows

$$\nabla_{<\mu}u_{\nu>} \equiv 2\nabla_{(\mu}u_{\nu)} - \frac{2}{3}\Delta_{\mu\nu}\nabla_\alpha u^\alpha \tag{3.34}$$

We get the expression for $\nabla_{(\mu}u_{\nu)}$ from above equation and putting it in the expression of s^μ we obtain

$$\partial_\mu s^\mu = \frac{1}{2T}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} + \frac{1}{T}\Pi\nabla_\alpha u^\alpha \geq 0 \tag{3.35}$$

We want this inequality to be fulfilled always. This can be ensured by making the terms on the left to be positive definite. So if we choose $\pi^{\mu\nu}$ and Π such that

$$\begin{aligned} \pi^{\mu\nu} &= \eta\nabla^{<\mu}u^{\nu>} \\ \Pi &= \xi\nabla_\alpha u^\alpha \end{aligned} \tag{3.36}$$

where ξ, η are positive constants. With these the viscous stress tensor becomes

$$\Pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>} + \xi\Delta^{\mu\nu}\nabla_\alpha u^\alpha \tag{3.37}$$

In the non relativistic limit the viscous stress tensor becomes that of Navier-Stokes equation and η, ξ are nothing but the shear and bulk viscosity coefficent. Equation 3.28 with this form of $\Pi^{\mu\nu}$ are called relativistic Navier Stokes Equation.

3.4 Acausality Problem of the relativistic Navier-Stokes equation.

Relativistic Navier-Stokes equation exhibits an unanticipated flaw which makes it unphysical. As a consequence of this equation we can have a frame of reference in which the effect can precede it's cause, which is not allowed for any physical situation. To see this we first consider the fluid to be in equillibrium. Then we perturb the system around this equillibrium and see the possible solutions for velocity of the fluid using Navier-Stokes equation. If it exceeds the speed of light in vacuum then the equation is acausal. On the other hand if the velocity of the fluid is bounded by some maximum value dependent on some parameters, then we say that the equation is causal.

Let us consider small perturbation of the energy density and fluid velocity in a system which was initially in equillibrium and at rest,

$$\begin{aligned} \epsilon &= \epsilon_0 + \delta\epsilon(t,x) \\ u^\mu &= (1,\vec{0}) + \delta u^\mu(t,x) \end{aligned} \tag{3.38}$$

Here we have considered only x-dependence of all the components of velocity perturbation and energy density perturbation for simplicity of the calculations. The evolution of spae time perturbation is given by the Navier-Stokes equation. We choose a particular direction $\alpha = y$, for which eqn. 3.28 becomes

$$(\epsilon + p)Du^y - \nabla^y p + \Delta^y_\nu \partial_\mu \Pi^{\mu\nu} = (\epsilon_0 + p_0)\partial_t \delta u^y + \partial_x \Pi^{xy} = O(\delta^2) \tag{3.39}$$

where

$$\Pi^{xy} = \eta(\nabla^x u^y + \nabla^y u^x) + \left(\xi - \frac{2}{3}\eta\right)\Delta^{xy}\nabla_\alpha u^\alpha = -\eta_0 \partial_x \delta u^y + O(\delta^2) \tag{3.40}$$

Now using the expression of Π^{xy} and neglecting terms of order $O(\delta^2)$ we have

$$\partial_t \delta u^y - \frac{\eta_0}{\epsilon_0 + p_0}\partial_x^2 \delta u^y = 0 \tag{3.41}$$

For the solution of above differential equation we use the ansatz

$$\delta u^y(t,x) = f_{\omega,k} e^{\omega t + ikx}$$

This solution is a plane wave with an exponential damping in time. It can be physically thought of as a plane wave with wave number k and amplitude decreases exponentially fast. Different wave number k corresponds to different modes of solution For this solution we have

$$\partial_t \delta u^y = \omega f_{\omega,k} e^{\omega t + ikx}$$

and

$$\partial_x^2 \delta u^y = -k^2 f_{\omega,k} e^{\omega t + ikx}$$

Putting these in the differential equation we have

$$\omega = \frac{\eta_0}{\epsilon_0 + p_0} k^2 \tag{3.42}$$

We can use this relation to find the group velocity v_g, given by

$$v_g(k) = \frac{d\omega}{dk} = 2\frac{\eta_0}{\epsilon_0 + p_0} k \tag{3.43}$$

Note that the group velocity is directly proportional to k. This implies that as k becomes larger then v_g also increases. For sufficiently large values of k, the velocity v_g can be greater than the speed of light. Anything travelling faster than the speed of light in one lorentz frame implies that there exist an inertial frame in which it is moving backward in time. Hence, the relativistic Navier-Stokes equation does is not a causal theory.

The problem in acausal theory is that it does not allow for free choice of initial conditions. Hydrodynamics is an initial value problem. If for some values of k we have time evolution in negative direction then we the initial values cannot be given freely and it is not possible to solve the Navier-Stokes eqiuation numerically.

3.5 Maxwell-Cattaneo Approach

There are different theories given to impose causality on the viscous fluid dynamics equations. One of the the approach as given by “Maxwell-Cattaneo law” is to introduce a term of time derivative of Π^{xy} with a factor of τ. The expression for Π^{xy} becomes

$$\tau \partial_t \Pi^{xy} + \Pi^{xy} = -\eta_0 \partial_x \delta u^y \tag{3.44}$$

or

$$\Pi^{xy} = -\eta_0 \partial_x \delta u^y - \tau \partial_t \Pi^{xy}$$

Again to check whether the theory is causal or not we find the group velocity using dispersion relation.

Again we have to solve

$$(\epsilon_0 + p_0)\partial_t \delta u^y + \partial_x \Pi^{xy} = 0 \tag{3.45}$$

Using eqn. 3.44 we have

$$(\epsilon_0 + p_0)\partial_t \delta u^y + \partial_x \left(-\eta_0 \partial_x \delta u^y - \tau \partial_t \Pi^{xy}\right) = 0$$

Again using the same expression of Π^{xy} in above equation we obtain

$$(\epsilon_0 + p_0)\partial_t \delta u^y + \partial_x \left(-\eta_0 \partial_x \delta u^y - \tau \partial_t \left(-\eta_0 \partial_x \delta u^y - \tau \partial_t \Pi^{xy}\right)\right) = 0$$

Now we use the expression $\Pi^{xy} = -\eta_0 \partial_x \delta u^y$ in above equation and using the same ansatz solution as before we get after some trivial manipulation,

$$\omega = \frac{\eta_0 k^2}{\epsilon_0 + p_0} \left(\frac{1 - (\omega\tau)^4}{1 - \omega\tau} \right)$$

Requiring that $\omega\tau << 1$ we get

$$\omega = \frac{\eta_0}{\epsilon_0 + p_0} \frac{k^2}{1 - \omega\tau}$$

Note that for $\tau = 0$, we revive the dispersion relation for relativistic Navier-Stokes equation. With the introduction of τ we get a factor of $1/(1 - \omega\tau)$.

Note that it is a quadratic equation in ω,

$$\omega^2 - \frac{1}{\tau}\omega + \frac{\eta_0}{\tau(\epsilon_0 + p_0)} k^2 = 0 \tag{3.46}$$

Solving this we get

$$\omega = \frac{1}{2} \left(\frac{1}{\tau} \pm \sqrt{\frac{1}{\tau^2} - \frac{4\eta_0 k^2}{(\epsilon_0 + p_0)\tau}} \right) \tag{3.47}$$

Differentiating the absolute value of ω with respect to k, and taking limit $k \to \infty$,

$$\lim_{k \to \infty} \frac{d|\omega|}{dk} = \sqrt{\frac{\eta_0}{(\epsilon_0 + p_0)\tau}}$$

Since v_g increases with k but tends to a limitting value we obtain the maximum velocity

$$v_g^{max} = \sqrt{\frac{\eta_0}{(\epsilon_0 + p_0)\tau}}$$

For suitable values of parameters in the expression of v_g^{max} the velocity can be limitted to be less than the speed of light. In fact, it is the case for some of the known fluids. Hence we conclude that Maxwell-Cattaneo law is an extension of relativistic Navier-Stokes equation that preserves causality. Even though it successfully ensures causality, we should note that it is not derived from any first principles. Rather it is just put by hand. Nevertheless, it's importance lies in the fact that it restores casuality of the viscous fluid dynamics equations.

3.6 Muller-Israel-Stewart Theory

In the covariant formulation of second law of thermodynamics $\partial_\mu s^\mu$, we had used the form of entropy current in equillibrium, $s^\mu = su^\mu$. But in a dissipative medium the fluid may not be in equillibrium. Hence we require entropy current

$$s^\mu \leq su^\mu$$

So we need to subtract some positive definite terms from the equillibrium term su^μ. Note that u^μ is the fluid variable here. For a higher order theory we need to add terms containing higher order derivatives of u^μ, such as $\partial_\nu u_\mu$. Also we require that the equillibrium surface to be convex for thermodynamic stability. There should not be a terms which will lead to point of inflexion in the entropy surface. So possible terms are even powers of odd derivative of u^μ and powers of even derivatives of u^μ. If we consider only the first derivative of u^μ, we need to have terms like $(\partial_\nu u_\mu)^2$ to the lowest order. A suitable choice is to have terms with Π^2 and $\pi^{\mu\nu}\pi_{\mu\nu}$, as Π and $\pi^{\mu\nu}$ are first derivatives of u^μ upto some factors. Satisfying these conditions and dimensions, the entropy current has to be of the form

$$s^\mu = su^\mu - \frac{\beta_0}{2T}u^\mu\Pi^2 - \frac{\beta_2}{2T}u^\mu\pi_{\alpha\beta}\pi^{\alpha\beta} + O(\Pi^3) \tag{3.48}$$

where β_0 and β_2 are coefficients that quantify the effect of the second order modifications of entropy current,

Again using eqn. 3.28 in the expression for increasing entropy we get

$$\begin{aligned}\partial_\mu s^\mu =& \frac{\pi^{\alpha\beta}}{2T}\left(\nabla_{<\alpha}u_{\beta>} - \Pi_{\alpha\beta}TD\left(\frac{\beta_2}{T}\right) - 2\beta_2 D\pi_{\alpha\beta} - \beta_2\pi_{\alpha\beta}\partial_\mu u^\mu\right)\\ &+ \frac{\Pi}{T}\left(\nabla_\alpha u^\alpha - \frac{1}{2}\Pi TD\left(\frac{\beta_0}{T}\right) - \beta_0 D\Pi - \frac{1}{2}\beta_0\Pi\partial_\mu u^\mu\right) \geq 0\end{aligned} \tag{3.49}$$

We want the terms which are getting subtracted to be positive definite. This can be ensured if

$$\begin{aligned}\pi_{\alpha\beta} &= \eta\left(\nabla_{<\alpha}u_{\beta>} - \Pi_{\alpha\beta}TD\left(\frac{\beta_2}{T}\right) - 2\beta_2 D\pi_{\alpha\beta} - \beta_2\pi_{\alpha\beta}\partial_\mu u^\mu\right)\\ \Pi &= \xi\left(\nabla_\alpha u^\alpha - \frac{1}{2}\Pi TD\left(\frac{\beta_0}{T}\right) - \beta_0 D\Pi - \frac{1}{2}\beta_0\Pi\partial_\mu u^\mu\right)\end{aligned} \tag{3.50}$$

with η and ξ as the shear and bulk viscosity coefficients. In the limit of $\beta_0, \beta_2 \to 0$ we obtain the corresponding equations in relativistic Navier Stokes equation.

Note that terms like $D\pi^{\alpha\beta}$ contains time derivatives of $\pi^{\alpha\beta}$ and hence we obtain terms similar to that in " Maxwell-Cattaneo law". So we identify the coefficients as proportional to a relaxation time

$$\begin{aligned}\beta_0 &= \frac{\tau_\Pi}{\xi} \\ \beta_2 &= \frac{\tau_\pi}{2\eta}\end{aligned} \tag{3.51}$$

The set of equations 3.28 with the form of $\Pi^{\alpha\beta}$ given by above two equations are known as " Muller-Israel-Stewart" equations.

Similar to previous sections, we study the casuality properties of the " Muller-Israel-Stewart " theory by considering a small perturbation around equillibrium. We use the same x dependent perturbation as we did in previous section. Keeping only the terms upto first order, eqn. 3.28 and eqn. 3.50 become the following set of simultaneous ddifferential equations

$$\begin{aligned}\partial_t\delta\epsilon + (\epsilon_0 + p_0)\partial_x\delta u^x &= 0 \\ (\epsilon_0 + p_0)\partial_t\delta u^x + \partial_x p + \partial_\mu\delta\Pi^{\mu x} &= 0 \\ (\epsilon_0 + p_0)\partial_t\delta u^y + \partial_\mu\delta\Pi^{\mu y} &= 0 \\ \delta\Pi^{\mu\nu} &= \delta\pi^{\mu\nu} + g^{\mu\nu}\delta\Pi \\ \delta\pi^{xx} + \tau_\pi\partial_t\delta\pi^{xx} &= -\frac{4}{3}\eta_0\partial_x\delta u^x \\ \delta\pi^{xy} + \tau_\pi\partial_t\delta\pi^{xy} &= -\eta_0\partial_x\delta u^y \\ \delta\Pi + \tau_\Pi\partial_t\delta\Pi &= \xi_0\partial_x\delta u^y\end{aligned} \tag{3.52}$$

To solve the abouve set of equations we use ansatz

$$\begin{aligned}\delta\epsilon &= e^{i\omega t - ikx}\delta\epsilon_{\omega,k} \\ \delta u^i &= e^{i\omega t - ikx}\delta u^i_{\omega,k} \\ \delta\pi^{\mu\nu} &= e^{i\omega t - ikx}\delta\pi^{\mu\nu}_{\omega,k} \\ \delta\Pi &= e^{i\omega t - ikx}\delta\Pi_{\omega,k}\end{aligned}$$

Using eqn. 3.6 in eqn. 3.52 we obtain folleing set of equations.

$$i\omega\delta\epsilon_{\omega,k} - ik(\epsilon_0 + p_0)\delta u^x_{\omega,k} = 0 \tag{3.53}$$

$$i\omega(\epsilon_0 + p_0)\delta u^x_{\omega,k} - ik\frac{dp}{d\epsilon}\delta\epsilon_{\omega,k} - ik\left(\frac{4}{3}\frac{ik\eta_0}{1 + i\omega\tau_\pi + \frac{ik\xi_0}{1+i\omega\tau_\Pi}}\right)\delta u^x_{\omega,k} = 0 \tag{3.54}$$

$$i\omega(\epsilon_0 + p_0)\delta^y_{\omega,k} - ik\left(\frac{ik\eta_0}{1+i\omega\tau_\pi}\right)\delta u^y_{\omega,k} = 0 \tag{3.55}$$

Note that eqn. 3.55 corresponds to Maxwell-Cattaneo result for the perturbation δu^y. So δu^y has an upper bound as given by "Maxwell-Cattaneo law".

Equations 3.53 and 3.54 relates the change in energy density and longitudinal (along x in our case) perturbation of fluid velocity δu^x. The longitudinal perturbation is commonly referred to as sound. The sound dispersion relation is obtained by using the expression of $\epsilon\omega, k$ from eqn. 3.53 in eqn. 3.54, which turns out to be

$$i\omega - i\frac{k^2}{\omega}\frac{dp}{d\epsilon} + k^2\left(\frac{4}{3}\frac{\eta_0}{\epsilon_0+p_0}\frac{1}{1+i\omega\tau_\pi} + \frac{\xi_0}{\epsilon_0+p_0}\frac{1}{1+i\omega\tau_\Pi}\right) = 0 \tag{3.56}$$

This equation can be solved for ω. The solution in the limit $\omega, k \to 0$ is given by

$$\begin{aligned}\omega = &\pm kc_s + ik^2\left(\frac{2}{3}\frac{\eta_0}{\epsilon_0+p_0} + \frac{1}{2}\frac{\xi_0}{\epsilon_0+p_0}\right)\\ &\mp \frac{k^3}{2c_s}\left[\left(\frac{2}{3}\frac{\eta_0}{\epsilon_0+p_0} + \frac{1}{2}\frac{\xi_0}{\epsilon_0+p_0}\right)^2 - 2c_s^2\left(\frac{2}{3}\frac{\eta_0}{\epsilon_0+p_0}\tau_\pi + \frac{1}{2}\frac{\xi_0}{\epsilon_0+p_0}\tau_\Pi\right)\right]\\ &+ O(k^4)\end{aligned} \tag{3.57}$$

where

$$c_s = \sqrt{\frac{dp}{d\epsilon}} \tag{3.58}$$

In order to understand the significance of c_s we take the limit $\omega, k \to 0$ in eqn. 3.56 and get

$$i\omega - i\frac{k^2}{\omega}\frac{dp}{d\epsilon} \approx 0 \tag{3.59}$$

or

$$\frac{\omega}{k} = \sqrt{\frac{dp}{d\epsilon}} \tag{3.60}$$

Hence $c_s = \omega/k$, the phase velocity of perturbation in the longitudinal(x) direction. Hence it is justified to be the sound waves in the fluid.

For large wavenumbers and frequencies $\omega, k \to \infty$, we use eqn. 3.56 can be simplified to

$$\omega^2 = k^2\left(c_s^2 + \frac{4}{3}\frac{\eta_0}{\tau_\pi(\epsilon_0+p_0)} + \frac{\xi_0}{\tau_\Pi(\epsilon_0+p_0)}\right) \tag{3.61}$$

From this we obtain a maximum limit of velocity

$$\lim_{k\to\infty}\frac{d\omega}{dk}=\sqrt{c_s^2+\frac{4}{3}\frac{\eta_0}{\tau_\pi(\epsilon_0+p_0)}+\frac{\xi_0}{\tau_\Pi(\epsilon_0+p_0)}} \tag{3.62}$$

Hence there is an upper limit to the velocity in the longitudinal mode. Let us denote this velocity by v_L^{max} and that obtained by for transverse mode is v_T^{max}. Hence we have an upper limit to the velocity v^{max}. So, " Muller-Israel-Stewart" theory, which is derived from second law of thermody is a relativistic theory of viscous hydrodynamics that obeys causality provided that the time τ_π and τ_Π are large enough to make the value of v_L^{max} less than the speed of light. The requirement that v_L^{max} to have an upper limit is more restrictive as it needs to restrict the domain of values taken by $c_s, \eta, \xi, \tau_\pi, \tau_\Pi$.

Even though it solves the problem of causality problem of relativistic Navier-Stokes equations, it is not yet a completely developed formalism. One may question the validity of assumption that the entropy current should be algebraic in hydrodynamic degrees of freedom. Also, this formalism does not provide a way to determine the value of τ_π and τ_Π. Further work in this field has to be done in order to answer these questions.

Chapter 4

Conclusion

The fluid dynamics equations in non-relativistic case are “continuity equation” and “ Euler equation” which are manifestations of conservation of mass and Newton’s second law respectively. In the relativistiv generalisation the equations are a consequence of conservation of energy momentum tesor for the fluids. In the non-relativistic limit we obtain the continuity equation and Euler equation from relativistic fluid equations. To take viscous effects into account an extra term $\Pi^{\mu\nu}$ is added to the energy momentum tensor for ideal fluid. Different theories prescribe different forms of $\Pi^{\mu\nu}$. The relativistic Navier-Stokes equation derives $\Pi^{\mu\nu}$ from the second law of thermodynamics $\partial_\mu u^\mu$ using the equillibrium expression for entropy current $s^\mu = su^\mu$. To test for causality, a small perturbation is introduced in energy density and fluid velocity around equillibrium state of the fluid, which has dependence only on one coordinate. The solution for evolution of the perturbation gives a dispersion relation from which velocity can ve calculated. It turns out that the velocity obtained by above prescription, in the case of relativistic Navier-Stokes equation, can take values greater than the speed of light. This is the acausality problem which makes this equation unphysical. So attempts were made to impose causality into the theory of viscous hydrodynamics. This is done by prescribing a form of $\Pi^{\mu\nu}$ which will lead to restoration of causality. In the Maxwell-Cattaneo approach an extra time derivative of $\Pi^{\mu\nu}$ was added to the expression of the same obtained by taking equillibrium entropy current expression. It turns out that the velocity obtained using this form of $\Pi^{\mu\nu}$ is bounded above. Hence, it constitutes a causal theory. But the extra term was just put “by hand” rather than any physical insight. The “Muller-Israel-Stewart theory” derives the form of $\Pi^{\mu\nu}$ by using the entropy current expression which is not in equillibrium, unlike the relativistic Navier-Stokes equation. The perturbative solutions now gave rise to two dispersion relations. The dispersion relation corresponding to that component of fluid velocity which is perpendicular to the axis on which perturbation depends, is of the form of Maxwell-Cattaneo type. This clearly has an upper limit of the corresponding velocity. The dispersion relation for the fluid velocity parallel to the axis on which perturbation depends, also gives an upper limit to the velocity. Sice an upper limit of the velocity is obtained, this theory respects causality.

Even though "Muller-Israel-Stewart theory" solves the causality problem of relativistic Navier-Stokes equations, it is not yet a completely developed formalism. One may question the validity of assumption that the entropy current should be algebraic in hydrodynamic degrees of freedom. Also, this formalism does not provide a way to determine the value of τ_π and τ_Π. Further work in this field has to be done in order to answer these questions. Also one may object the method to test for causality of a theory. The method we used was a sufficient condition to check for acausality. But to test for causality one must consider large perturbations also. The claim we made for causality is for small perturbations. Techniques of calculation need to be developed to test casuality of a theory for large perturbations.

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References

[1] P. Romatschke, New Developments in Relativistic Viscous Hydrodynamics, Int. J. Mod. Phys.E19:1-53,2010, `arXiv:0902.3663v3`

[2] Landau and Lifshitz, “Fluid Mechanics, Course on theoretical Physics”, Volume 6