Sixth Semester Project

Acausality Problem In Relativistic Hydrodynamics.

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Non-relativistic Ideal Fluid Dynamics

The degrees of freedom for an ideal, neutral, one-component fluid are are the fluid velocity $\vec{v}(t,\vec{x})$, the pressure $p(t,\vec{x})$ and the fluid mass density $\rho(t,\vec{x})$. These variables define the state of the fluid which are linked by the fluid dynamic equations, that is, the Euler equation

$$\partial_t \vec{v} + (\vec{v}\cdot\vec{\partial})\vec{v} = -\frac{1}{\rho}\vec{\partial}p \tag{1}$$

and the Continuity equation

$$\partial_t \rho + \rho\vec{\partial}\cdot\vec{v} + \vec{v}\cdot\vec{\partial}\rho = 0 \tag{2}$$

These equations have to be supplemented by an equation of state $p = p(\rho)$ to close the system.

Non-relativistic Viscous Fluid Dynamics

For non-ideal fluids, where dissipation can occur, the Euler equation generalises to the "Navier-Stokes equation".

$$\frac{\partial v^i}{\partial t} + v^k\frac{\partial v^i}{\partial x^k} = -\frac{1}{\rho}\frac{\partial p}{\partial x^i} - \frac{1}{\rho}\frac{\partial \Pi^{ki}}{\partial x^k} \tag{3}$$

$$\Pi^{ki} = -\eta\left(\frac{\partial v^i}{\partial x^k} + \frac{\partial v^k}{\partial x^i} - \frac{2}{3}\delta^{ki}\frac{\partial v^l}{\partial x^l}\right) - \xi\delta^{ki}\frac{\partial v^l}{\partial x^l} \tag{4}$$

where $i = 1,2,3$ denotes three space directions. The viscous stress tensor Π^{ki} contains the coefficients for shear viscosity, η, and bulk viscosity, ξ, which are positive and velocity independent. The non-relativistic Navier-Stokes equation is well tested and reliable, so any successful theory of relativistic hydrodynamics should reduce to it in the non-relativistic limit.

Relativistic Ideal Fluid Dynamics

The relativistic fluid dynamics equations are a consequence of conservation of energy-momentum tensor $T^{\mu\nu}$ for a relativistic fluid. The energy-momentum tensor of an ideal relativistic fluid is given by

$$T_0^{\mu\nu} = \epsilon u^\mu u^\nu - p\Delta^{\mu\nu} \tag{5}$$

where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$. Here ϵ is the energy density, u^μ is the four velocity, and $g^{\mu\nu}$ is the metric tensor $diag(+,-,-,-)$. As a consequence of conservation of energy-momentum tensor $\partial_\mu T_0^{\mu\nu}$ one obtains

$$D\epsilon + (\epsilon + p)\partial_\mu u^\mu = 0 \tag{6}$$

$$(\epsilon + p)Du^\alpha - \nabla^\alpha p = 0 \tag{7}$$

where $D \equiv u^\mu\partial_\mu$ and $\nabla^\alpha \equiv \Delta^{\mu\nu}\partial_\mu$. These are fundamental equations for a relativistic ideal fluid. In the non-relativistic limit they reduce to continuity equation and Euler equation.

Relativistic Viscous Fluid Dynamics

The relativistic Navier-Stokes equation

To include the viscosity effects we add another term to the non-relativistic ideal fluid energy-momentum tensor

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu} \tag{8}$$

Again the conservation of energy-momentum tensor leads to

$$D\epsilon + (\epsilon + p)\partial_\mu u^\mu - \Pi^{\mu\nu}\partial_{(\mu}u_{\nu)} = 0 \tag{9}$$

$$(\epsilon + p)Du^\alpha - \nabla^\alpha p + \Delta_\nu^\alpha\partial_\mu\Pi^{\mu\nu} = 0 \tag{10}$$

where $A_{(\mu}B_{\nu)} = \frac{1}{2}(A_\mu B_\nu + B_\nu A_\mu)$. For different theories of viscous fluid dynamics, one gets different forms of $\Pi^{\mu\nu}$. The second law of thermodynamics $\partial_\mu s^\mu$ is used to specify the form of $\Pi^{\mu\nu}$. For relativistic Navier-Stokes equation the relation $s^\mu = su^\mu$ is used. $\Pi^{\mu\nu}$ is split into a traceless part $\pi^{\mu\nu}$ and a term with non-vanishing trace $\Delta^{\mu\nu}\Pi$. With the notation for traceless part of $\partial_{(\mu}u_{\nu)}$

$$\nabla_{<\mu}u_{\nu>} \equiv 2\nabla_{(\mu}u_{\nu)} - \frac{2}{3}\Delta_{\mu\nu}\nabla_\alpha u^\alpha \tag{11}$$

the form of $\Pi^{\mu\nu}$ obtained is

$$\pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>}, \qquad \Pi = \xi\nabla_\alpha u^\alpha \tag{12}$$

The Acausality Problem

To study causality properties, an x-dependent perturbation in energy density ($\epsilon = \epsilon_0 + \delta\epsilon(t,x)$) and fluid velocity ($u^\mu = (1,\vec{0}) + \delta u^\mu(t,x)$) is introduced. For $\alpha = y$ one obtains

$$(\epsilon_0 + p_0)\partial_t\delta u^y + \partial_x\Pi^{xy} \approx 0 \tag{13}$$

where

$$\Pi^{xy} \approx -\eta_0\partial_x\delta u^y \tag{14}$$

$$\partial_t\delta u^y - \frac{\eta_0}{\epsilon_0 + p_0}\partial_x^2\delta u^y = 0 \tag{15}$$

Ansatz $\delta u^y(t,x) = f_{\omega,k}e^{\omega t + ikx}$ gives the dispersion relation

$$\omega = \frac{\eta_0}{\epsilon_0 + p_0}k^2 \tag{16}$$

The group velocity obtained by this relation is

$$v_g(k) = \frac{d\omega}{dk} = 2\frac{\eta_0}{\epsilon_0 + p_0}k \tag{17}$$

For sufficiently large values of k, the velocity v_g can be greater than the speed of light. This is the problem of Acausality.

Maxwell-Cattaneo Approach

A new term $\tau\partial_t\Pi^{xy}$ is introduced so that

$$\Pi^{xy} = -\eta_0\partial_x\delta u^y - \tau\partial_t\Pi^{xy} \tag{18}$$

The dispersion relation obtained using this gives a group velocity which has an upper bound

$$v_T^{max} = \sqrt{\frac{\eta_0}{(\epsilon_0 + p_0)\tau}} \tag{19}$$

So, this is an extension of relativistic Navier-Stokes equation that preserves causality. Even though it successfully ensures causality, it is not derived from any first principles. Rather it is just put by hand.

Muller-Israel-Stewart Approach

Unlike the relativistic Navier-Stokes equation, in this case the equilibrium expression is not used for entropy current. Instead the for of s^μ is

$$s^\mu = su^\mu - \frac{\beta_0}{2T}u^\mu\Pi^2 - \frac{\beta_2}{2T}u^\mu\pi_{\alpha\beta}\pi^{\alpha\beta} \tag{20}$$

Using this in second law of thermodynamics, and taking appropriate approximations we get a set of differential equations which is solved by using plane wave ansatz. When solved we get a relation similar to that of Maxwell Cattaneo in transverse direction (perpendicular to x), and two other equations which gives dispersion relation for longitudinal direction (along x). The velocity obtained using these dispersion relations have an upper limit

$$v_L^{max} = \sqrt{c_s^2 + \frac{4}{3}\frac{\eta_0}{\tau_\pi(\epsilon_0 + p_0)} + \frac{\xi_0}{\tau_\Pi(\epsilon_0 + p_0)}} \tag{21}$$

where $c_s = \sqrt{dp/d\epsilon}$ and it is identified that $\beta_0 = \tau_\Pi/\xi$, $\beta_2 = \tau_\pi/2\eta$. Since there is an upper limit to the allowed velocity it constitutes a causal theory.

Further work

Even though the problem of causality problem of relativistic Navier-Stokes equations, a completely developed formalism is not available. The maximum velocity limit was established only for small perturbations. One may question the validity of assumption that the entropy current should be algebraic in hydrodynamic degrees of freedom. Also, this formalism does not provide a way to determine the value of τ_π and τ_Π. Further work in this field has to be done in order to answer these questions.

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